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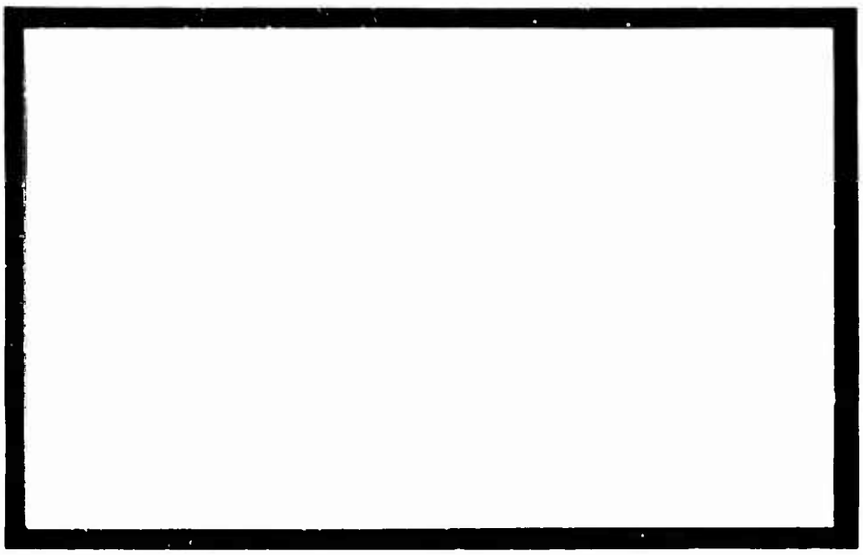

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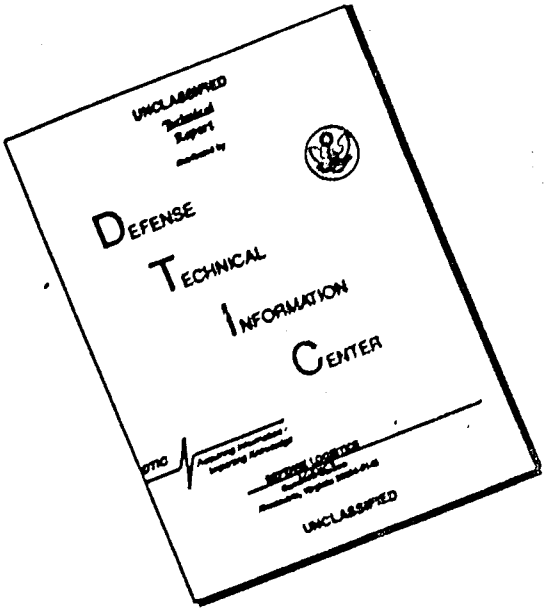

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SEARCH AND INFORMATION THEORY *

Part of Final Report On

Stochastic Processes

In Certain Naval Operations

by

Bernard O. Koopman

UNCLASSIFIED

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SEARCH AND INFORMATION THEORY

B. O. Koopman

1. Introduction

Ever since the mid-nineteen-forties when the theories of information and of search became subjects of general interest, attempts have been made to apply the theory of information to problems of search. These have proved disappointing; neither the formulas nor the concepts of the former theory have found a place in clarifying the problems of the latter. It has seemed to the present author that this fact is a natural consequence of a fundamental difference in the subject-matter of the two theories: in search, the geometry (in the sense of positions, distance, areas, etc.) is an essential factor of the operation--in the elementary act of detection is to select a position and look near it. In the classical theory of information, on the other hand, no attention is paid to such metric matters, the ideas being confined to dichotomies: the elementary act is to ascertain in which of two subsets of a given set (e.g., of states of a system) the actual object (or state) belongs; and the geometrical shape or extent of the subsets has no necessary connection with the operation.

The purpose of the present investigation is to turn the question around, and to seek, not what applications information theory has to search, but what light search can throw on a broadened conception of information theory. The key idea is to start with the notion of the elementary detecting operation and then to see what kind of quantitative measure of information can be obtained by its optimally repeated use We shall use

the language of search; but in order to bring out the essential simplicity and generality of the ideas, we shall put the definition in a somewhat general form.

In search there are <u>two</u> places where probability can enter: in the probability distribution of the target's possible positions before the search; and in the conditional probability that the searching operation succeed in detecting it--given that it is present at the place searched.

Regarding the first, it will be assumed henceforth that the target is in an unknown position x on a certain set X of possible positions, but that its probabilities p of being in the various positions in X are known. If there are only a finite number of positions in X, all their probabilities are given; if X has a continuum of possible points, the probability density is given; etc.[1]

Regarding the second--the conditional probability of detecting--it will be assumed that there is an <u>elementary detecting operation</u>, repetitive in nature, and capable of answering certain of the searcher's questions concerning the target's position. This operation may only succeed with a certain (known) probability of giving the answer; but when it does, it is truthful: we are not considering the possibility of false contacts.

Concerning the elementary detecting operations, we set up a <u>schema</u> Σ of these operations in such a way that, by carrying them out in a succession depending on the results as they develop, the target's position is finally found--either exactly or within a pre-stated degree of accuracy. The maximum number of operations needed may be finite, as when X has but n positions and the elementary operation consists in asking whether the target is in some subset A of X or not--the question always being answered. Or the number of operations may be one integer on one occasion and another on a

second--every positive integer representing a possibility. But in every case the schema determines a random variable N_Σ, the number of detecting operations up to target localization (with the accuracy stated). Since N_Σ is non-negative, it has an expected value EN_Σ, finite or not.

These notions will be given concrete illustration by the examples examined later. For the moment we merely observe that each performance of the elementary detecting operation is thought of as representing a liability or cost expenditure (in units of money, time lost, degree of exposure to danger, etc.). Therefore, EN_Σ is a "bad" quantity which we seek to minimize by our choice of Σ.

Against this background we lay down the following:

DEFINITION. The quantitative measure of uncertainty $U[p]$ in the probability distribution p (i.e., in (X, S, p)) is the minimum of EN_Σ--or its greatest lower bound--for all possible choices of schemata Σ. Further, we define the information as $I[p] = C - U[p]$ the constant C being so chosen that $I[p] \geq 0$.

2. Operational Compatibility

The ideas involved in the target's probability distributions and their combinations lead to no conceptual difficulties. But those which concern the elementary detecting operations and their probabilistic combinations (with each other and with the probability distributions) give rise to hitherto unsuspected difficulties, which were first brought into evidence by J. M. Dobbie.[2]

There is in fact a parting of the ways--according to whether the elementary detecting operation has a material effect upon the situation it is intended to examine, or whether it merely increases the searcher's knowledge without altering anything else in the world.

Only with the advent of modern quantum mechanics, in the nineteen twenties, has the basic issue involved here been explicitly identified. The "principle of indeterminacy" (better, "restricted accuracy") is illustrated by the impossibility of experimentally determining both the position and the conjugate momentum of an elementary particle beyond a limited accuracy. The reason is that, within the framework of this theory, statements about position and momentum can only be statements about the outcomes of position and momentum observations, whose actual performance involves a mutual interference.

Rather slowly it is being realized that this issue is by no means confined to phenomena at the level of the elementary particles of physics. An example from biology and from military search will illustrate what is basically involved.

Suppose that a hitherto unknown mutant of a laboratory rat is to be examined for resistance to two toxins, A and B. If X is the length of time of survival after the untreated rat is exposed to A, and if Y is the corresponding quantity for B, it is evident that X and Y are defined by incompatible operations. If large numbers of such rats become available, one could measure the averages X and Y, and so, by the law of large numbers, evaluate their expected values, EX and EY. But the expected value theorem

$$E(X + Y) = EX + EY$$

would be false--not because the laws of probability are violated, but because

the symbol A + Y, as a chance variable measured by the class of one-rat experiments, involves an operational contradiction. More primitively, if α is the statement "the rat dies within an hour of sole exposure to A" and β is the corresponding statement for B, we cannot apply the theorems of probability to the logical combinations αβ, α + β, etc.--not because probability is wrong, but because α and β are incompatible events: αβ and α + β are meaningless according to the definition of α and of β as one-rat events.

The German use of search-receivers against radar in World War II gives a second illustration of the point: After a first radar search without result, a second search of the same region has a probability of success affected, not only by Bayesian reasoning based on the negative result of the first, but by the fact that the hostile target may have detected the presence of the searcher and taken measures of concealment (e.g., submerged, if a submarine): The first act and essentially change the conditions of the second.

J. M. Dobbie's example is the case of search for an object dropped on a sandy beach, when a first search may have the physical effect of accidentally covering it with sand.[2]

Only with the basic postulate that all the events and random variables considered together in a probability system are compatible[3]--i.e., definable by non-interfering physical acts of observation--does the situation exist for which the laws of probability are conventionally stated.[4]

Such compatibility shall be assumed in what follows.

3. Specific Cases

The concepts of the last two sections are illustrated by cases falling into two types.

In the first type, the elementary detecting operation has a unit probability of success. For example, in the definite range law of search, it is assumed that the target is detected if and only if it is within a circle of "detection range R" of the searcher. In case of the effect of target aspect, the circular region may be replaced by one of a different shape; and similarly in non-symmetrical looking (e.g., anisotropic array gain).

In these cases, we can say that the class S of subsets of X (ref.[1]) contains the circles, or their modifications; and that the elementary detecting operation is that of seeing whether the target is or is not in one of these special members of S.

A more flexible situation is that in which any member of S--i.e., any operationally meaningful subset of X--can be selected; and then the presence or absence of the target in it determined by the detecting observation. We shall call this the case of unrestricted dichotomy.

In all these cases we have defined the elementary operations, but have not yet examined the result of their successive performance, nor considered the construction of Σ-schemata which will guide the strategy of our search-to-localization. It is here that the question of compatibility must be faced.

In the case of the definite range law, let the first elementary operation consist in placing the observer at the point x_o. The probability of detecting the target is the probability of its being within range R of x_o, i.e., the integral over this circle of the probability density p(x). But suppose the result is negative; can we say that on a second performance of the elementary operation, centered at another point x_1, the probability

of detection is found by applying the same formula to the probability density $P_o(x)$ obtained from $p(x)$ by Bayes' formula? The answer is in the affirmative only if the target's location is unaffected by the first operation--i.e., in the case of compatibility of all the relevant events.

A similar statement applies to the other examples just given of this first type of case--in particular, the case of unrestrained dichotomy. And in view of our present assumption of compatibility, Bayes' formula will be applied.

In the second type of elementary detecting operation, the conditional probability of detecting, given the target at the place of observation, may be less than unity. Here the question of compatibility applies not only to the law of change of the a posteriori probability distribution, but to the conditional probability of success of later detecting operations. With our assumptions, the former is Bayesian as stated before. The latter is evidently given by the survival probability formula for repeated independent trials: if P_o is the probability of detection of one elementary operation (given that the target is in the place searched), $1 - (1 - P_o)^n$ is the probability of detection by n repetitions. This can be written as $1 - e^{-\mu n}$ where $\mu = -\log(1 - P_o) > 0$. In this form, n may be regarded as the number of units of searching effort.

A more general and important situation is that in which there is a continuum of elementary detecting operations, measured by a parameter u which expresses the intensity of search, or amount of searching effort, directed at a given reference point x_o. Let $P_o(u)$ be the conditional probability of detecting the target, given that it is at x_o and that the effort u is applied: i.e., make the broad but still restrictive assumption that when (x_o, u) are given, the conditional detection probability is determined.

If u is first applied and then v, the probability of detection of the whole operation is $P_o(u + v)$. Now since we are assuming compatibility of the elementary operations, we can regard the latter as equivalent to the conjunction of the two former (e.g., first u, then v). And since the target is given at x_o the probabilities are independent. Therefore, by elementary probability, the complementary probabilities $Q_o(u) = 1 - P_o(u)$, etc., satisfy the functional equation $Q_o(u + v) = Q_o(u)\ Q_o(v)$. On adjoining to this the obvious fact that $Q_o(u)$ decreases as u increases (the more effort, the more chance of detection), we can derive rigorously the solution

$$P_o(u) = 1 - e^{-\mu_o u} = 1 - e^{-\mu(x_o)u} .$$

where μ_o is positive. This is, of course, formula of random search.[5] It cannot be too strongly emphasized that this derivation would be wrong without the assumption of compatibility.

The remainder of this paper will apply the definition of information of $\mathcal{I}$ 1 first to the case of unrestricted dichotomy; second, to that of random search. The former will take us into contact with classical information theory, while the latter will lead into a new area, and throw light on the process of surveillance.

4. Unrestricted Dichotomy in the Finite Case

We assume that the set X contains only the finite number n of possible positions and that the elementary detecting operation consists in subdividing any subset X^1 of X (possibly = X) into any two complementary subsets ($X^1 = X_1^1 + X_2^1$) and then finding which one contains the target.

If we wish to repeat this process until the position of the target is ascertained, we must first decide on the decomposition of X into X_1 and X_2. If the first operation gives that the target is in X_1, we must then decide how to divide X_1 in two. Similarly, if the target is given in X_2. This process must be repeated until the target is found. Nothing prevents our making all possible choices of dichotomies in advance. Thus we are led to the schema Σ, consisting of a complete system of branching dichotomies, as follows (using + to denote the set sum of two mutually exclusive sets, and "order" to mean the number of subdivision-and-question procedures):

First order: $X = X_1 + X_2$

Second order: $X_1 = X_{11} + X_{12}$, $X_2 = X_{21} + X_{22}$

Third order: $X_{11} = X_{111} + X_{112}, \ldots, X_{22} = X_{221} + X_{222}$

. . . . . .

In writing this out it is understood that in a given horizontal line, we stop a dichotomy of any subset containing just one element; and on the other hand, we always push the dichotomization until this is the case for every subset. The result is a schema Σ appropriate to the present problem. It can be represented graphically as a "tree", always branching in two, with branches finally terminating in points, n in all, corresponding with the number of positions in the searched set X.

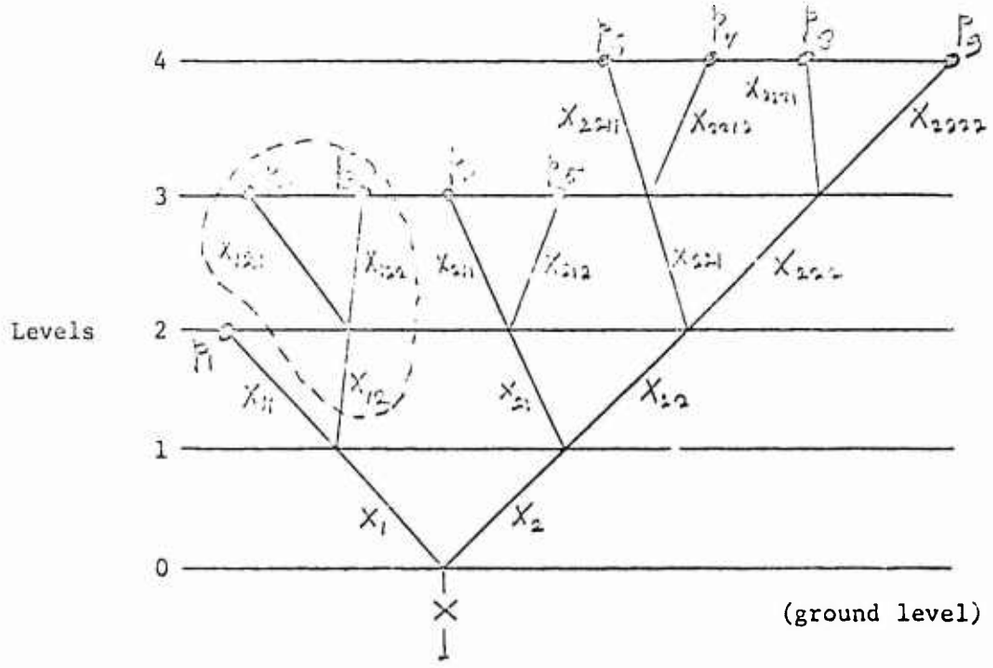

Fig. 1 A Schema Σ tree with n = 9, h = 4.

Let us think of the terminations and the branch-points of the tree as grouped into fixed levels corresponding to the orders of the dichotomies they represent, their heights i above ground being the corresponding number of units of length. Then i will run from 1 to h, the last order of dichotomy--the "height of the tree" (= 0 when there is no dichotomy: only one $p_i \neq 0$). Finally, mark the terminating points with the symbols $p_1, \ldots, p_n$ for the given probabilities that the target be in the positions $x_1, \ldots x_m$. Thus we obtain a graphical representation, of the type shown in Fig. 1, of a schema Σ. We are interested in the corresponding expected value EN_Σ of dichotomies needed to reach the target.

It is convenient to pass from this graphical representation to a mechanical one: Imagine the tree as a weightless rigid frame of branches, to the n terminating ends of which are attached particles of masses $p_1, \ldots, p_n$, as exemplified in Fig. 1. Then the formulas of elementary mechanics

show that EN_Σ is the height of the center of gravity of this loaded tree. Hence, the optimum schema, which gives U[p] by minimizing EN_Σ, is the tree of lowest center of gravity bearing the given $p_1,\ldots,p_n$ in the way described above.

The operations of "branch-interchange" transform a tree bearing $p_1,\ldots,p_n$ into another such tree: to characterize the tree of lowest center of gravity we must examine their effect on the position of this point. For this purpose, a precise notation is useful:

Consider the line segments between the levels i-1 and i: each one defines a branch, composed of all the segments joined to it above (directly or indirectly) and all the terminating particles they bear (cf, the branch enclosed by the dotted line in Fig. 1). At one extreme, the branch could reduce to a single line segment terminated by one particle. Every line segment will be identified by two indices (i,j), where i is the height of its upper extremity (i = 1,...,h) and j is a second identifying index, running from 1 to n_i, the number of segments whose tops are at the level i. Clearly, $n_1 = 1$; while for i = 2,...,h, we have $n_i \geq 2$. The same two indices identify a branch--the one determined by the line segment of these indices. Let w_{ij} denote the weight of the branch (i, j); i.e., its total load of terminating particles. Thus, in the indicated branch of Fig. 1 we have $w_{2,\ 2} = p_2 + p_3$.

A basic tree operation is the interchange of two disjoint branches, e.g., (i, j) and (i^1, j^1), having no part in common. This, of course, includes the interchange of the particles themselves. The useful and self-evident fact is that for the tree of lowest center of gravity, every such interchange raises it or leaves its height unchanged.

If the branches (i, j) and (i', j') are at the same level (i.e., $i' = i$), their interchange produces no change in the center of gravity. If, on the other hand, $j' = j + 1$ $(\geq n)$, the interchange will raise the center of gravity or lower it accordingly as $w_{i,j} > w_{i+1,j'}$ or $w_{i,j} < w_{i+1,j'}$.

By the use of this principle, the following two facts are easily established:

A. If all the probabilities are equal ($p_1 = \ldots = p_n = 1/n$), the optimum schema Σ is obtained by finding the two non-negative integers h, k for which

$$n = 2^h - k, \quad k < 2^{h-1};$$

then, if $k > 0$, placing k particles at the level $h - 1$ (in a complete tree with 2^{h-1} terminations at that level); and finally, at each of the $2^{h-1} - k$ unused points, placing pairs of branches, to whose $2(2^{h-1} - k) = 2^h - 2k = n - k$ terminations are attached the remaining $n - k$ particles. Using this schema, we obtain

$$U = \frac{1}{n}[(h-1)k + h(n-k)]$$

$$= h - \frac{k}{n}$$

This may be compared with the diadic entropy

$$H = \sum_{i=1}^{-n} p_i \log_2 p_i = \log_2 n$$

of the present distribution. We have

$$U - H = h - \frac{k}{n} - \log_2 n$$

$$= \log_2(n+k) - \frac{k}{n} - \log_2 n$$

$$= \log_2\left(1 + \frac{k}{n}\right) - \frac{k}{n},$$

which, as will be seen later, is non-negative. Since $0 \leq \frac{k}{n} < 1$, we have the asymptotic formula

$$U = H(1 + \theta_n)$$

where $\theta_n \to 0$ as $n \to \infty$.

1. If every probability in $(p_1, \ldots, p_n)$ is a positive integral power of $\frac{1}{2}$, U equals the diadic entropy $U = H$. For in this case a tree can be constructed in an obvious manner so that each probability $\left[\frac{1}{2}\right]^i$ is the mass of a particle of height i--and no branch interchange can lower the resulting center of gravity.

2. In the most general case of probability $(p_1, \ldots, p_n)$, we have the inequality $U \geq H$.

To prove this, let Σ be any schema, optimum or not, and in the corresponding tree, redesignate the probabilities with two indices as p_{ij}, where, if there are no particles of height i, $p_{ij} = 0$; if there are s_i such particles, j is an identifying index going from 1 to s_i. Clearly

$$U \leq \sum_{i=1}^{h} i \sum_{j=1}^{s_i} p_{ij}$$

$$H = -\sum_{i=1}^{h} \sum_{j=1}^{s_i} p_{ij} \log_2 p_{ij} .$$

Now consider a second tree, of the same branches as the given one, but whose particles at height i have the masses $\left[\frac{1}{2}\right]^i$. That this is a legitimate distribution, i.e., that the sum of the masses is always unity, is shown by the following mass counting process: each particle at height $i < h$ and mass $\left[\frac{1}{2}\right]^i$ can be replaced by two particles of masses $\left[\frac{1}{2}\right]^{i+1}$ at height $i + 1$ joined to the original position by two line segments, without

altering the total mass. Continuing in this way, the tree is replaced by one of equal weight and having all its particles of mass $\left\lceil \frac{1}{2} \right\rceil$ at height h: there being 2^h such particles, its total mass is unity.

If p^1_{ij} is the mass of the particle replacing p_{ij} in the original tree, we have by a formula of convex functions in information theory,

$$\sum_{i=1}^{h} \sum_{j=1}^{s_i} p_{ij} \log_2 \frac{p_{ij}}{p^1_{ij}} \geq 0 ,$$

($= 0$ if and only if every $p^1_{ij} = p_{ij}$) .

Hence

$$H = \sum_{i=1}^{h} \sum_{j=1}^{s_i} p_{ij} \log_2 p_{ij} \geq \sum_{i=1}^{h} \sum_{j=1}^{s_i} p_{ij} \log_2 p^1_{ij}$$

$$= \sum_{i=1}^{h} \sum_{j=1}^{s_i} p_{ij}(-i)$$

$$= -\sum_{i=1}^{h} s_i \Sigma p_{ij}$$

This, combined with the definition of U and the fact that $H = -\Sigma p_i \log p_i$ is independent of any particular tree, leads to the desired result $U \geq H$.

Thus, the close relation--but not identity--between the search-theoretic operational definition of uncertainty and information theory, and the diadic entropy and its negative, the quantity of information, are shown. One might describe the situation by saying that only exceptionally can the full amount of classical information be extracted by the dichotomy searching process.

5. Unrestricted Dichotomy in Continuum

When the set X of possible positions x of the target is a curve, surface, or higher dimensional continuum, on which the given probability density p(x) is essentially continuous, the expected number of dichotomy operations up to exact localization is, of course, infinite, and the previous ideas need to be modified--and indeed the practical problem of search shows that the difficulties are the result of a refinement irrelevant to the problem. For, once the target has been localized in a sufficiently small region, it is as good as found: after all, the "position x of the target" means the position of a reference point in the target; and if the latter is is a solid, its physical dimensions will extend about this reference point. Or if our "target" is not a solid but, e.g., a radiation, and we are searching for its "position" in a space of such characteristics of radiation as frequency, polarization, direction, etc., once these are "boxed up" in a sufficiently small region, the practical problem is solved.

If X is a finite line segment and if target localization within a small distance Δx is sufficient, we have but to cover X by n non-overlapping segments (x_{i-1}, x_i) of length Δx, and then to apply the methods of §4 to the n probabilities obtained by integrating p(x) over each of these sub-segments.

In this case, diadic entropy becomes (using the law of the mean for integrals, etc.)

$$H = -\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} p(x)\,dx \cdot \log_2 \int_{x_{i-1}}^{x_i} p(x)\,dx$$

$$= -\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} p(x)\,dx \cdot [\log_2 p(\bar{x}_i) + \log_2 \Delta x]\,dx$$

$$= -\int_X p(x)\ \log_2 p(x)\ dx - \log_2 \Delta x + Z(\Delta x)$$

$$= H[p(x)] - \log_2 \Delta x + Z(\Delta x)$$

where $Z(\Delta x)$ denotes a quantity approaching zero with Δx, and $H[p(x)]$ is the diadic entropy of a continuous distribution. This means that (neglecting $Z(\Delta x)$) it is not $H[p(x)]$ that is the greatest lower bound of the uncertainty U, but this quantity plus $-\log_2 \Delta x$, which <u>becomes infinite</u> as $\Delta x \to 0$. Therefore, our search-theoretic definition of uncertainty requires modification, since in its previous form its value will be crucially dependent on the criterion of accuracy Δx.

In order to secure a more intrinsic conception of "search uncertainty"--one less conditioned by the value of Δx--we may proceed as follows:

When Δx is given (in addition to X and hence n), we might reasonably think of its "<u>standard effect</u>" as the uncertainty in the special case when p(x) is <u>most unfavorable</u>; i.e., as the maximum expected number of dichotomies, when--keeping Δx, etc. fixed--the results of making all possible choices of p(x) are compared. Evident reasoning based on the results A, B, C of §4 shows that the maximum in question will occur when the n intervals Δx are of equal probabilities, i.e., when p(x) is constant; and that then the uncertainty differs by less than unity from the value $\log_2 n = \log_2 L - \log_2 \Delta x$, where L is the length of the interval X. Moreover, then as $\Delta x \to 0$ (i.e., $n \to \infty$), the minimum is asymptotic to the above expression (their ratio approaching unity).

The algebraic excess of the actual uncertainty for the given p(x) over its greatest possible value is bounded below by a quantity asymptotic to

$$H[p(x)] - \log_2 L \ ,$$

as $\Delta x \to 0$. When p(x) = constant, this uncertainty is itself asymptotic to the above quantity, which is itself independent of Δx.

It is noted that, if q(x) is the uniform distribution over X, i.e., if q(x) = 1/L, the above expression can be written as -G[p, q], where

$$G[p, q] = \int_X p(x) \log_2 \frac{p(x)}{q(x)} \, dx \ .$$

Now this G[p, q], which will re-appear later, is the fundamental two-distribution information (the negative of the "cross-entropy") introduced in 1950 by a number of authors,[6] and representing intuitively the increase in information concerning the position of x in X, conferred by any datum leading one to replace the probabilities q(x) by the new ones p(x). It is shown to be between 0 and $+\infty$, equal to the former if and only if p(x) and q(x) represent the same probability distribution.

It is necessary to consider cases that, in two respects, go beyond the simple one just discussed. The first generalization maintains a one-dimensional X, but requires that the interval Δx of acceptable accuracy be different at different positions, as when a second type of search will follow the first and will have a power of detection which varies with position. For example, if a definite range law is to be used, its range could vary with varying visibility or background noise from point to point. Let two values

x' and x'' ($x'<x''$) be given; the degree of inaccuracy of a search giving that the target x is in the interval (x', x''), which in the previous case was its length $\Delta x = x'' - x'$, is now a more general function of x', x''; i.e., $F(x', x'')$.

Clearly, if $x'''>x''$ is a third point, it seems natural to regard the inaccuracies as additive; i.e.,

$$F(x', x''') = F(x', x'') + F(x'', x''') .$$

But this means that, for sets composed of adjacent intervals, the inaccuracy F is an additive set function--obviously non-negative. By a reasoning that is as old as the calculus (although re-phrased in all precision and generality in the modern theory of integration), we can at once conclude that under all conditions of physical interest, the limit

$$f(x) = \lim_{x' \to x} \frac{f(x, x')}{x'-x}$$

exists, and that $F(x', x'')$ is its integral from x' to x''.

We now make a change of variable, setting $y = \phi(x)$, where $\phi(x)$ has a continuous positive derivative $\phi'(x)$. This will transform the interval $(a \leqq x \leqq b)$ into another one, $(\bar{a} \leqq y \leqq \bar{b})$. On the other hand, since $p(x)$ and $f(x)$ are densities, the quantities $p(x)\,dx$ and $f(x)\,dx$ are preserved in value. Hence $p(x)$ and $f(x)$ are replaced, respectively by

$$\bar{p}(y) = \bar{p}(\phi(x)) = p(x)/\phi'(x)$$

$$\bar{f}(y) = \bar{f}(\phi(x)) = f(x)/\phi'(x) .$$

We next select the particular change of variables function $\phi(x)$ so that $\bar{f}(y) = 1$ and $\bar{a} = 0$; i.e., take

$$y = \phi(x) = \int_o^x f(x)\, dx \ .$$

This means that since

$$\int_{\bar{a}}^{\bar{b}} \bar{f}(y)\, dy = \int_a^b f(x)\, dx = F(b, a) \ ,$$

we obtain, for the length of the interval $(\bar{a}, \bar{b})$,

$$L = F(b, a) \ ;$$

and further,

$$\bar{p}(y) = p(x)/f(x)$$

But this change of variables leads from our general problem back into the earlier one, now applied to the set Y of the values of the variable y, upon which set X has been mapped. The reduced diadic entropy expression

$$H[\bar{p}(y)] - \log_2 L = -\int_{\bar{a}}^{\bar{b}} \bar{p}(y) \log_2 \bar{p}(y)\, dy - \log_2 L \ .$$

This becomes, on substitution, etc.,

$$-\int_a^b \frac{p(x)}{f(x)} \left(\log_2 \frac{p(x)}{f(x)}\right) f(x)\, dx - \log_2 F(b, a)$$

$$= -\int p(x) \log_2 \frac{p(x)}{q(x)}\, dx \ , = -G[p, q]$$

where we have written

$$q(x) = \frac{f(x)}{\int_a^b f(x)\, dx} \ .$$

Note that the $q(x)$ as defined has the properties of a probability density.

The final result is again that the excess of the expected number of dichotomies needed for detection to the accuracy F(x, x') over its maximum value has -G[p, q] as a lower bound, asymptotically as $\Delta x \to 0$.

Our second generalization is to the case of a higher dimensional X. In view of the detailed considerations of the previous cases, it may be permitted to treat this case rather summarily. We shall suppose that two non-negative additive set-functions P(A) and F(A) are given; the first is the probability, before the dichotomy search, that the target be in the set A; the second is the degree of inaccuracy of the datum that the target has been ascertained to be in A. By a change of variables, X can in practical cases be mapped on a set Y so that F(A) is replaced by the "Extent" (length, area, volume, etc.) of A. When the total value F(X) is finite, we may "normalize" F(A) to unity, i.e., replace it by the probability set-function

$$Q(A) = \frac{F(A)}{F(X)} \quad .$$

Then the greatest lower bound of the excess of the expected excess of the number of dichotomies for localization to F-accuracy (or Q-accuracy) over its maximum is asymptotic to -G[P, Q], where

$$G[P, Q] = \int \log_2\left(\frac{dP}{dQ}\right) \cdot dP = \int p(x) \log_2 \frac{p(x)}{q(x)} \, dx \ ,$$

in which p(x) and q(x) are the density functions corresponding to P(A) and Q(A)--or, better, $\frac{dP}{dQ}$ is the derivative of the set function P with respect to Q. The final result is the following, which we state here in terms of the search-information, as defined in §1 (The negative of the uncertainty, and taken as non-negative.)

The search information in the distribution P, relative to the accuracy standard Q, is bounded below by a quantity asymptotic, as $Q(\Delta x) \to 0$, to the general diadic information $G[P, Q]$.

6. The Case of Random Search

Search, in the original Naval sense of the term, is conducted by operations that do not fall into the class of the dichotomy, but which bring to bear upon a certain locality a determined degree of effort. As explained in the second half of §3, when the elementary searching operation is the expenditure of the degree of effort u in the locality of the point x, and when all such efforts there are compatible, the conditional probability of success is $1 - e^{-\mu u}$, where μ may differ for different points x. This is the law of random search[5], and will form the basis for the present application of the conception of search information, formulated in §1.

When X is k-dimensional and dx is a k-dimensional element of volume, the intensity $u = u(x)$ of search at x must, when integrated over X, give (with the additivity assumptions underlying the discussion) the total searching effort U applied to the whole of X:

$$\int_X u(x)\, dx = U \ . \tag{6.1}$$

The problem of the optimum distribution of this given quantity of total effort, when the target's probability density $p(x)$ is given, can be interpreted as follows:[5]

Maximize the probability of detection

$$P[U, p] = \int_X p(x)[1 - e^{-\mu(x)\, u(x)}]\, dx \tag{6.2}$$

subject to the equation (6.1) as well as to the inequality

$$u(x) \geqq 0 \tag{6.3}$$

The solution is given in the reference[5]. A point of view of sequential optimization has recently been developed by J. M. Dobbie[2], particularly in cases in which our assumption of compatibility is not made. Before going further, we must be more precise about how the quantities are measured.

The total searching effort U shall be measured in units of k-dimensional "volume searched" (area, if X is a region of the surface of the ocean). Hence, the same will be true of $u(x)\, dx$, the element of integration in (6.1); and $u(x)$ will be dimensionless if and only if the coordinates are lengths. But in every case, a change of variables of integration must leave $u(x)\, dx$ invariant, and hence multiply $u(x)$ by the Jacobian of the original coordinates of x with respect to the new coordinates. Since $p(x)\, dx$ is both invariant and dimensionless (being a probability), $p(x)$ is of dimension depending on that of the coordinates $\mathbf{x}$ ($[L^{-k}]$, if they are lengths), and changing as $u(x)$ does under changes of variables of integration. Since, furthermore, the integral in (6.2) is a probability, and hence dimensionless, the quantity $\mu(x)\, u(x)$ must also be dimensionless; therefore, the dimensions of $\mu(x)$ must be the reciprocals of those of $u(x)$. Finally, since $\mu(x)\, u(x)$ must, by similar reasoning based on (6.2), remain invariant under a change of variables of integration, such a change must multiply

$\mu(x)$ by the reciprocal of the Jacobian of the old coordinates with respect to the new: i.e., by the Jacobian of the new with respect to the old.

We shall use these facts to standardize our expressions. Let us introduce the new function

$$\phi(x) = \mu(x)\, u(x) \tag{6.4}$$

It is invariant under change of variables of integration, and is dimensionless. In terms of this, (6.1) becomes

$$\int_X \phi(x) \frac{dx}{\mu(x)} = U \ .$$

Now make a change of variables of integration, selecting the new variables, $\bar{x}$, so that in the invariant expression

$$\frac{dx}{\mu(x)} = \frac{d\bar{x}}{\bar{\mu}(\bar{x})} \ ,$$

the $\bar{\mu}(\bar{x}) = 1$. In other words, introduce such variables $\bar{x}$ that the Jacobian

$$\frac{\partial(x)}{\partial(\bar{x})} = \frac{\partial(x_1, \ldots, x_k)}{\partial(\bar{x}_1, \ldots, \bar{x}_k)} = \mu(x) = \mu(x_1, \ldots, x_k) \ .$$

Such a selection is, clearly, always possible.

As a result, the optimization problem corresponding to (6.1), (6.2), (6.3), is replaced by the one treated at the outset of the reference[5] (replacing ϕ by U) viz., of finding that function $\phi(x)$, among the class of functions satisfying

$$\int_X \phi(x)\, dx = U \ , \qquad \phi(x) \geq 0 \ , \tag{6.5}$$

which maximizes

$$P[U, p] = \int_X p(x)\{1 - e^{-\phi(x)}\}\, dx \ . \tag{6.6}$$

In all that follows, the notation and results of that reference will be used implicityly (log denoting the natural logarithm).

We have to consider the expected "number of elementary searching operations", EN_Σ , up to and including detection. Here, the number N_Σ is evidently to be interpreted as the amount of effort, ϕ_Σ , used up in the search up to the moment of detection. To obtain its maximum expected value, the schema Σ must schedule each unit of effort optimally as a sequential process. But it is known[5] that this is the same thing as to have the total amount of effort up to any point optimally scheduled. Henceforth, the schema Σ will denote such an optimum scheduling; and (dropping this subscript), ϕ shall denote the random variable defined as the quantity of effort just used up at the moment when detection is made--assuming optimum scheduling throughout. Thus, ϕ is a chance variable having a probability distribution of taking on various values U.

With this definition, the "uncertainty"--the U[p] of §1--is given by the expected value formula

$$U[p] = E\phi = \int_0^\infty U \, d \, \text{prob}[\phi \leq U] \ . \tag{6.7}$$

Using the notation of the reference[5] (log being the natural logarithm), we introduce the non-negative variable λ, the family of subsets A_λ of X parameterized by λ, defined as the set of points for which $p(x) \geq \lambda$:

(6.8) $$A_\lambda = [x \mid p(x) \geq \lambda] \ ;$$

and also the k-dimensional volume $v = v(\lambda)$ of A_λ (area, if X is a plane):

(6.9) $$v(\lambda) = \int_{A_\lambda} dx \ .$$

Since A_λ increases in content with decreasing λ, coinciding with X when $\lambda = 0$, $v(\lambda)$ increases monotonically with decreasing λ.

Similarly, the probability that the target be in A_λ, given by

(6.10) $$\pi(\lambda) = \int_{A_\lambda} p(x)\, dx \ ,$$

is a function of λ, increasing monotonically from 0 to 1 as λ decreases from $+\infty$ to 0. Furthermore, for any value of λ for which $v(\lambda)$ has a derivative $V'(\lambda)$, $\pi(\lambda)$ does likewise, and

(6.11) $$\pi'(\lambda) = \lambda V'(\lambda) \ .$$

This is easily shown by the elementary application of definitions and of the law of the mean. A generalization of such a relation to a jump relation $\delta\pi(\lambda) = \lambda\delta v(\lambda)$ can be made, but will not be used here, since almost always in applications the behavior of $p(x)$ is such that (6.11) applies at all points. Finally, on setting

(6.12) $$S(\lambda) = \int_{A_\lambda} \log\frac{p(x)}{\lambda}\, dx = \int_{A_\lambda} \log p(x)\, dx - v(\lambda) \log\lambda \ ,$$

We recall that it was shown[5] that $S(\lambda)$ increases monotonically from 0 to $+\infty$ as λ decreases from $+\infty$ to zero, and does so continuously, taking on each intermediate value exactly once. This means that any discontinuities in the two terms on the right in (6.12) cancel.

In these terms, we can state the results of reference[5] for the optimum searching scheduling as follows:

For each given total available effort U, determine the unique λ satisfying the equation:

$$S(\lambda) = U \ . \tag{6.13}$$

On the corresponding set A_λ , apply the intensity of searching effort

$$\phi(x) = \log\frac{p(x)}{\lambda} \ ; \tag{6.14}$$

On the remaining set, $X - A_\lambda$, apply no searching intensity: $\phi(x) = 0$. Thus, the maximum probability of detection--i.e., the probability, when the effort is optimally programmed, as described, is

$$P(U) = \pi(\lambda) - \lambda\nu(\lambda) \ . \tag{6.15}$$

Clearly, this may also be described as the probability that the total searching used up at the moment of detection, i.e., Φ, shall not exceed U:

$$P(U) = \text{prob}[\Phi \leq U] \ .$$

Accordingly, (6.7) gives

$$U[p] = E\phi = \int_0^\infty U \, d[\pi(\lambda) - \lambda\nu(\lambda)] \ . \tag{6.16}$$

Here the right-hand expression must, in the most general case, be understood as a Stieltjes integral. But in all practical applications (6.11) applies and we can then reduce it to the formula

$$U[p] = \int_0^\infty U\nu(\lambda) \, dx \tag{6.17}$$

(The minus sign is absorbed in the reversal of order of integration on the interval $0 \leqq \lambda < +\infty$, etc.)

7. Application to the Normal Distribution

One of the most common situations in search is that in which the target's given probability distribution is normal ; i.e., when (after diagonalizing),

$$(7.1) \qquad p(x) = \frac{1}{\sigma_1 \ldots \sigma_k} \frac{1}{(2\pi)^{k/2}} \exp - 1/2 \sum_{i=1}^{k} \frac{(x_i - a_i)^2}{\sigma_i^2} \quad .$$

Not only does such a distribution occur when the target's position can be regarded as due to the accumulation of a large number of small random displacements or errors of navigation or observation, so that the central limit theorem leads to (7.1); but in many cases of much more complicated probability density (uni-modal or otherwise), when the probabilities in all but certain places are small enough to be neglected, while in other places they are peaked enough to make normal law expressions acceptable approximations.

Let us apply the formulas and methods of §6 to this case, taking our axes at the point (a) as origin--i.e., replacing $x_i - a_i$ by x_i . Then we have for A_λ the k-dimensional ellipsoidal region on which $p(x) \geqq \lambda$, bounded by the k-dimensional ellipsoid of equation

$$(7.2) \qquad \frac{x_1^2}{\sigma_1^2} + \ldots + \frac{x_k^2}{\sigma_k^2} = -\log \lambda^2 \sigma_1^2 \ldots \sigma_k^2 (2\pi)^k$$

$$= w^2 \quad .$$

The k-dimensional volume of this k-ellipsoid is found by standard processes of k-dimensional integration (changing variables to $y_i = x_i/\sigma_i$ and thus reducing to hyperspherical integration). We obtain

$$v(\lambda) = \sigma_1 \ldots \sigma_k \frac{(\sqrt{\pi})^k}{\Gamma(\frac{k}{2} + 1)} w^k \quad . \tag{7.3}$$

To apply (6.17), we have to express U in terms of λ by means of (6.13) and (6.12). Using (7.1), with the w as defined in (7.2), we obtain:

$$\log p(x) = \log\lambda + \frac{w^2}{2} - (1/2)\left(\frac{x_1^2}{\sigma_1^2} + \ldots + \frac{x_k^2}{\sigma_k^2}\right) ;$$

whence, on transposing the first term and integrating over A_λ,

$$S(\lambda) = \frac{w^2}{2}\left[v(\lambda) - \sigma_1 \ldots \sigma_k \frac{(\sqrt{\pi})^k w^k}{\Gamma(\frac{k}{2} + 2)} \frac{k}{k+2}\right] .$$

$$= \frac{w^2}{2} \sigma_1 \ldots \sigma_k \frac{(\sqrt{\pi})^k w^k}{\Gamma(\frac{k}{2} + 2)} = \frac{w^2}{k+1} v(\lambda) .$$

This value is substituted for U in (6.17), while the $d\lambda$ is replaced by the value obtained from the second equation in (7.2). On making these substitutions, absorbing a minus sign in the process of reversing the order of the limits of integration when the variable of integration λ is replaced by w, and, finally, using elementary properties of the gamma function, we obtain for the "uncertainty" U[p] in the present probability distribution:

$$U[p] = 2\sigma_1 \cdots \sigma_k (\sqrt{2}\pi)^k \frac{k!}{\Gamma(\frac{k}{2}+1)^2} \tag{7.4}$$

$$= 2v[w=\sqrt{2}] \cdot \frac{k!}{(\frac{k}{2}+1)}$$

In the case of the plane, $k = 2$, and our uncertainty $U[p] = 4v[w=\sqrt{2}]$. Here $v[w = \sqrt{2}]$ is the area of the ellipse of semi-axes $\sigma_1\sqrt{2}$, $\sigma_2\sqrt{2}$; it is called the "localization area" in the modern theory of surface search. It is such that the probability that the target is in it is $1 - \frac{1}{e}$.

More generally, the search uncertainty is proportional to the product of the k standard deviations, the constant of proportionality being explicitly expressed, through (7.4), in terms of the number of dimensions in question.

The application of the notion that the search "information" is the negative of the uncertainty, which was appropriate in the case of dichotomies, seems less appropriate in the present case, since it would be a negative number, becoming negatively infinite as any of the standard diviations increase without limit. Perhaps a more appropriate rendering of the concept would be the <u>reciprocal</u> of the uncertainty. This would be proportional to the height of the highest point of the normal distribution, a quantity long used as a measure of <u>precision</u>, in the application to the theory of measurement.

NOTES

1. In the technical language of the modern theory, we have a probability space (X, S, p), where S is the class of subsets A of X in which it is physically meaningful to say that the target can lie (the "measurable subsets") and p(A) their probabilities. Cf. P. R. Halmos Measure Theory (D. Van Nostrand Co., Inc., 1950) Chapter IX.

2. Cf.a forthcoming paper by J. M. Dobbie in Operations Research. Cf. also, J. M. Dobbie, "Search Theory: A Sequential Approach", Naval Research Logistics Quarterly, 10, (1963), pp. 331, 332.

3. Only in quantum mechanics has this nation been systemmatically exploited; there it is connected with the family of commuting observables. Cf. J. van Newmann, Mathematische Grundlagen der Quantenmechanik (Springer, Berlin, 1932).

4. Note the precise wording of this statement. We are not saying that the law of probability would be wrong when applied to incompatible events, but that, when precisely stated, their hypotheses would not be met in such cases. Cf. B. O. Koopman, Quantum Theory and the Foundations of Probability. Conference on Applied Mathematics, Sponsored by the Courant Institute (NYU) and ONR, McGraw-Hill, Co., 1955.

5. B. O. Koopman,"Theory of Search II, III", Operations Research, Vol. 4, No. 5, October 1956, p. 519; and Vol. 5, No. 5, October 1957, pp. 613-626.

6. For references and a development of this concept, see, e.g., S. Kullback, Information Theory and Statistics (N. Y., Wiley, 1959).

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THE LOGICAL BASIS OF COMBAT SIMULATION

by

Bernard O. Koopman

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THE LOGICAL BASIS OF COMBAT SIMULATION

B. O. Koopman

1. The Two Aspects of a Military Operation

The evaluation of planned weapon systems or of proposed tactics must be based, in last analysis, upon advantages foreseen in combat: in battles not yet fought. But in the real world, such events are of a complexity--both of kind and of number of combining factors--as to obscure the relationships of cause and effect. Yet if physicists can draw quantitative conclusions regarding the properties of matter in spite of the inconceivable complexity of its detailed molecular motions, we may hope to do likewise--if we learn to look in the right direction--in the study of combat. The present work examines the various methods that have been used for this purpose (analytic models, machine simulation, Monte Carlo, etc.) with the object of discovering their basic common principle.

On turning attention, not to methods or models, but to the military actions themselves, the most striking fact is their bivalence: their character both of an evolving physical system, and of an unfolding set of plans, intentions, reasoning and counter-reasoning of the men engaged in the action, the commanders. The two aspects must be examined separately before they can be comprehended together in the full military operation.

Moreover, for the evaluation of weapon systems and tactics, it is the physical behavior that is emphasized. How can this be separated from the human side without mutilating a whole that is greater than the sum of its parts and so losing the meaning of both of them?

Two methods have been used with some rational basis for studying the physical aspect of combat without the complications of the human one: the method of standardized decisions; and the method of minimax of game theory. They are most clearly explained in the context of the succeeding section. In concluding this one, we merely remark that the nearest to a systematic method for examining combat in its mental or human aspect is--in addition to the study of history itself--use of the war game, as carried out in staff colleges.

2. The System and Its States

Basic to any scientific examination of nature is the concept of the system: the set of interacting things considered. In a military action, the system is the totality of men and weapons involved, together with their environment: the medium in which the action occurs and which effects its course. And equally fundamental is the concept of the set of states that the system can be in, just one at any given time. Thus, in a duel between two aircraft, the system is the pair of aircraft, their weapons and equipment and personnel, and the air in which they are flying, including gravitational and electromagnetic fields. In a submarine attack on a convoy, the system is the set of vessels involved, their men and equipment, and the sea and air in which the action takes place. In each case, the state of the system includes its

physical state: positions and velocities of the units, condition of armaments, data-gathering status, and all the meteorological specifications. But how far into the mental state of the commanders must one go in defining the "state" of the system? This can only be settled by asking a second question, that of the evolution of the state of the system with the passage of time.

Classical physics has traditionally considered that the state of a system is only adequately described if, once the state is given, all later states are determined: Given any two similar systems in the same initial states, all their later states will be the same--provided that their environmental influences (external forces) continue the same. Thus, in Newtonian mechanics, the full and exact knowledge of the positions and velocities of the parts of a material system determine its whole future motion. But it is only in the simplest military operations that such an order of determinateness exists.

In far more cases, it is not feasible so to specify the state of a system that its subsequent evolution is determined. What is far more common is to have only statistical determinateness: in a large number of similar systems starting in the same state, the same proportion will go into any given later state. This is the situation in statistical mechanics, and in the more fully developed parts of operations research. It is the most that one may expect in combat operations.

This brings out the role of the human decision-maker in the evolution of the system: the most obvious lesson of history and common sense is that under the same conditions different commanders often make different decisions.

pointed that this in itself is enough to rule out any strict determinism, we have, nevertheless, to ask whether this unpredictability of decision could be encompassed in a statistical determinateness. In the context of the present study, this is a purely practical question; can we say that, when the system reaches such and such a state, we can specify sensibly the probabilities that the commanders will make the various conceivable decisions?

It is submitted that any such assignment of probabilities to decisions is unrealistic as a prediction of the future, but realistic (if it reflects a consensus of experienced commanders) as leading to a model serving to eliminate weapon systems or tactics. Usually a definite decision, rather than probabilities of various decisions, is formulated for each set of circumstances; i.e., for each state of our system. To reach such a consensus may involve an extensive discussion if several commanders differ. It may even require listing two different opinions when agreement cannot be reached, each of which is used on two performances of the game--rather than a "toss up" in a single one. Thus, the decision-maker's contribution to the indeterminateness is removed.

Another common method for removing humanly-caused indeterminateness is to assume, first, that at each stage of the action, each commander has a stated degree of knowledge concerning the other, assumes that the latter will always try to do him the maximum harm, and then picks his own course of action so as to minimize it. This is the minimax convention of game theory. When applied sequentially, it leads to differential games. Rightly employed, it gives a useful indicator in evaluations; it can never be relied on to predict the future.

From this point on, we shall assume that the human variability has been removed, and shall examine what remains: the statistically determinate evolution of the military system.

3. The Basic Stochastic Process

Having reached the stage at which the physical evolution of the system can be studied in itself, it is necessary to put the matter precisely. Let S be the system under consideration, and let X be the set of all its possible states, the latter being denoted by such lower-case letters as x, x', y, etc. In simple situations--or after simplifying approximations--X may contain only a finite number of states; but in general, the number will be infinite, and of a more or less high and complicated order. We shall, however, use the symbol Σ to denote summation over all its states, even in the cases where this may actually be an integration, $\int_X \ldots dx$, possibly of a very general nature (e.g., Radon-Stieltjes, etc.).

Let two states x, x' (distinct or not) in X be given. In virtue of the statistical determinateness, there is a definite probability $a(x, t ; x', t')$ (possibly = 0) that, if the system is in state x at time t, it will be in state x' at the later time $t'(>t)$. This is the <u>probability of the transition</u>

$$(x, t) \rightarrow (x', t')$$

and will be denoted by $a(x, t ; x', t')$. It is evidently a conditional probability; explicitly:

$a(t, x ; t', x') = P[S$ <u>in state</u> x' <u>at</u> $t' \mid S$ <u>in state</u> x <u>at</u> $t]$.

Evidently, if the values of the transition probabilities $\pi(x, t\ ;\ x', t')$ were all known, the probabilities of every outcome of the battle would be known--and this for every assumed starting state.

Thus, the whole problem of the quantitative study of military operations is that of finding the transition probabilities, from knowledge that can reasonably be obtained. We shall see how all the standard analytic models, Monte Carlo simulations, etc., fit into this scheme. In every case, what is obtained directly, by transcribing into probabilities the physical knowledge of the system and how it is operated, are the elementary transition probabilities: those for a short increment of time $\Delta t = t' - t$ --or, more exactly, transition rates. Furthermore, a recurrence relationship is always made use of (too often with insufficient justification), by means of which the general transition probabilties can be "built-up" (therefore, computed) from the elementary ones.

The system S with its set X of states and their transition probabilities constitute, in technical language, a stochastic process--the fundamental stochastic process of the military engagement in question. The computation of the transition probabilties reposes on the basic relations of the theory of stochastic processes, concerning which an abundant literature exists.*

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*Feller, "An Introduction to Probability and Its Applications", Vol. II. Wiley, New York, 1966.

Cox and Miller, "The Theory of Stochastic Processes". Wiley, New York, 1965.

Doob, "Stochastic Processes", Wiley, New York, 1953.

Loève, "Probability Theory", Van Nostrand, 1963.

Hille, "Functional Analysis and Semi-Groups", American Mathematical Society Colloquium Publications, Vol. XXXI.

We shall give the methods in outline (written in the simple notation Σ, as explained above). But before this can be done, one basic issue must be faced.

4. The Markovian Assumption

Suppose given three successive epochs, $t < t' < t''$ and two states, x and x'', and consider the transition

$$(x, t) \to (x'', t'')$$

which has the probability $a(x, t ; x'', t'')$. Since this transition occurs by going through some intermediate state, say x', at the intermediate time t', i.e., since the event $(x, t) \to (x'', t'')$ is the event stated as follows:

$$\text{for some } x' \text{ of } X: \quad (x, t) \to (x', t') \text{ and } (x', t') \to (x'', t'')$$

we should be able to express the transition probability $a(x, t; x'', t'')$ in terms of those of the intermediate transitions, by applying the laws of compound and total probability to the latter. In general, however, this does not mean that $a(x, t ; x'', t'')$ can be expressed in terms of $a(x, t ; x', t')$ and $a(x', t'; x'', t'')$, but rather of the former and the more complicated conditional probability

$$a(x, t ; x', t' ; x'', t'') = P[S \text{ in } x'' \text{ at } t'' | S \text{ in } x' \text{ at } t' \text{ and } S \text{ in } x \text{ at } t]$$

by the formula

$$(4.1) \quad a(x, t; x'', t'') = \sum_{x'} a(x, t; x', t')\, a(x, t; x', t'; x'', t'')$$

In actual mathematical models or machine simulations, this general formula (4.1) is never used, but is replaced by the formula*

$$(4.2) \qquad a(x, t; x'', t'') = \sum_{x'} a(x, t; x', t')\, a(x', t'; x'', t'')$$

which, as it is applied, it is tantamount to the assumption that

$$(4.3) \qquad P[S \text{ in } x'' \text{ at } t'' \mid S \text{ in } x' \text{ at } t' \text{ and } S \text{ in } x \text{ at } t] = P[S \text{ in } x'' \text{ at } t'' \mid S \text{ in } x' \text{ at } t']$$

This is the Markoff assumption and makes our stochastic process a Markoff process. Then (4.2) becomes valid, and is indeed the well-known Chapman-Kolmogorov equation, dominating the theory--and hence the practice--of the basic stochastic process of the present type of operation. While details will be given later in concrete context, it may be remarked already that in the case that the number of states in X is finite, the right-hand side of (4.2) is a matrix product.

Under what conditions is the Markov assumption (4.3) justified? In other words, when does the specification of the state x' at t' give such complete knowledge regarding the transitions from x' that any further data anterior to t' (e.g., that at $t<t'$ it was in state x) contributes nothing to the probabilities in question? In a general way, we may say that when there is a material factor in the situation that remains partly unknown after the state x' at t' has been specified, but concerning which we may draw inferences from knowledge of the previous history of the system, then the Markov assumption is not justified.

*This replacement--and even the formulas themselves--are too often not recognized explicitly in simulations, but can be discovered as implicite substrata underlying the concrete procedures of the numerical operations.

Such a failure of the Markov property is commonly produced by the attempt to simplify a treatment by an injudicious condensation of several different states into a single one. An example would be in sonar detection, when the "state" of the system does not sufficiently specify the acoustic condition of the water in which it takes place: the more detections are made (i.e., the more the transitions from states of noncontact to contact) the more the evidence that the acoustic conditions are favorable, and hence the greater the probability of further detections. The loaded die is another example--the oftener it shows ace, the more the chance that it will show ace again: the whole past influences probability predictions of the future.

Of course, when methods of computer simulation are made in the usual way and hence depend for their validity on the Markov property, but when this does not apply, for reasons such as those just set forth, the numerical results, however realistic they may appear, are without logical basis--at least until they are proved to give an acceptable degree of approximation. The act of simplifying and still retaining the Markovian character--as well as operational realism—is an art as well as a science. Success is more apt to be achieved by limiting the objective of the study to the answer of a precise question rather than a diffuse multitude.

5. Transition Rates

Having postulated that the basic stochastic process of our operation is Markovian, we shall now outline the theory and practical application of the methods for obtaining the operationally important transition probabilities (or the corresponding expected values, etc.) from the elementary transition

probabilities--those applying to such short intervals of time $\Delta t = t' - t$ that the large aspects of the system remain unchanged except by quantities of the order of Δt at most during Δt.

Any transition probability $a(x, t; x', t')$ has certain obvious properties resulting from the fact that it is a probability; thus

$$(5.1) \qquad a(x, t; x', t') \geq 0 \qquad \sum_x a(x, t; x', t') = 1 ,$$

the second formula expressing the principle of total probability. Further, since no transition has any probability of occurring in a zero time interval $a(x, t; x', t) = 0$ if $x' \neq x$. By continuity, $a(x, t; x', t') \to 0$ as $t' \to t$ $(t' > t)$, when $x' \neq x$. When X is a finite set, we can say that $a(x, t; x, t) = 1$, and that $a(x, t; x, t') \to 1$ as $t' > t$. In the general case, we have to use the symbolism of the Dirac delta function $\delta(x' - x)$ (the Kroneker delta in the discrete case), and write

$$a(x, t; x', t') \to \delta(x' - x) \quad \text{as } t \to t'$$

meaning that for any continuous function $f(x)$,

$$\sum_x f(x)\, a(x, t; x', t') \to f(x')$$

$$\sum_x a(x, t; x', t')\, f(x') \to f(x)$$

as $t \to t'$ $(t' > t)$.

In all the actual physical and operational cases, more can be assumed: it will always be true that, for small $t' - t > 0$, the two members of the preceding relation differ by quantities of the order of $t' - t$; and in fact a derivative exists. The most convenient way of writing this

[illegible] $= t' - t > 0$:

(5.2) [illegible] $b(x, x', t)\,\Delta t + [\Delta t]$

where [illegible] will be used [illegible] the generic expression for a quantity [illegible] that $[\Delta t]/\Delta t \to 0$ (uniformly in x, x', etc.) as $\Delta t \to 0$ ($\Delta t > 0$). For reasons that will appear below, $b(x, x'$ [illegible] the transition rate.

[illegible] has the properties [illegible] in (4.2), [illegible] that the transition rate is [illegible] second equation in (5.1) and [illegible] the x' sum of transition rates vanishes. [illegible]

(5.3) [illegible] $\sum_{x'} b(x, x', t) = 0$.

This shows that when X is [illegible] $b(x, x', t)$ is a matrix, the diagonal terms are minus the sum of the corresponding nondiagonal ones (in the same row). [illegible] fact in waiting-line problems.

The above relations will now be applied to the Chapman-Kolmogorov equation (4.2), with the result that two differential equations will be obtained, the so-called "forward" and "backward" equations.

First, write (5.2) in the form

$$a(x', t'; x'', t' + \Delta t) = \delta(x'' - x') + b(x', x'', t')\,\Delta t + [\Delta t]$$

and make this substitution in (4.2) (with $t'' = t' + \Delta t$). Obvious calculations

[illegible] $t') = \sum_{x'} a(x, t; x', t') \, b(x', x'', t')$

[illegible] we set $t' = t + \Delta t$ and substitute ([illegible]) directly into [illegible]

[illegible] $(x, x', t) \, a(x', t; x'', t'')$.

[illegible]—which, in spite [illegible] are in general highly [illegible] in matrix form in the [illegible] x, x', x'' are thought of [illegible] the matrices

$A(t, t') = \| a(x, t; x', t') \|$,

$B(t) = \| b(x, x', t) \|$.

[illegible] our equations bec[illegible] writing t' in place of t'' in the second)

(5.4)' $\quad \frac{\partial A(t, t')}{\partial t'} = A(t, t') \, B(t')$

$- \frac{\partial A(t, t')}{\partial t} = B(t) \, A(t, t')$

It may also be observed that the Chapman-Kolmogorov equation (4.2) assumes the form

([illegible]) $\quad A(t, t'') = A(t, t') \, A(t', t'')$.

In this finite case, the problem of finding the general transition probabilities (here, the matrix $A(t, t')$) from the more elementary transition rates (the matrix $B(t)$) is the old problem of solving a system of first order homogeneous linear differential equations: for each row of the matrix $A(t, t')$ is, for fixed t (e.g., $t = 0$) a system of functions satisfying the differential system (5.4)', whose initial values (when $t' = t$; e.g., $t' = 0$) are determined by (5.2). The different rows in $A(t, t')$ form a complete set of linearly independent solutions. The resulting solutions automatically satisfy (5.5)'; and vice versa, by solving (5.5)' (looking now at the columns in $A(t, t')$) we get the solution of (5.4)'. In the finite case considered, these solutions are uniquely determined by the differential equations, and because of (5.3), they automatically satisfy (5.1).

The situation naturally becomes far more complicated for infinite x, not merely with respect to the approximate numerical calculation of the solutions, but even with regard to such basic questions as existence and uniqueness, and the deduction of (5.1) from (5.3). Even in the case when X is a discrete but infinite set, as occurs in many waiting line problems and birth-death processes of practical interest, uniqueness and property (5.1) may fail. For further discussion, the references given above should be consulted.

An essential simplification occurs when the general conditions leading to the transitions remain the same throughout the period of time considered in the whole action. Then the probability of a transition

$$(x, t) \quad \rightarrow \quad (x', t')$$

will not depend on the starting time t but only on the elapsed time t'-t, so we may write

$$a(x, t; x', t') = a(x, x'; t' - t) \quad . \tag{5.7}$$

We then say that we have a stationary transition Markov process.

With such a process, the Chapman-Kolmogorov equation may be written in the form (with $t_1 = t' - t$, $t_2 = t'' - t'$)

$$a(x, x''; t_1 + t_2) = \sum_{x'} a(x, x'; t_1)\, a(x', x''; t_2) \tag{5.8}$$

while (5.3) becomes

$$a(x, x'; \Delta t) = \delta(x' - x) + b(x, x')\, \Delta t + [\Delta t].$$

These two equations express the semi-group property of our transition probabilities (Cf. E. Hille, l.c.). Equations (5.4) and (5.5) may now be written (with t replacing t' - t and t'' - t) as

$$\frac{\partial}{\partial t} a(x, x'', t) = \sum_{x'} a(x, x', t)\, b(x', x'') \tag{5.9}$$

$$\frac{\partial}{\partial t} a(x, x''; t) = \sum_{x'} b(x, x')\, a(x', x''; t) \tag{5.10}$$

The matrix equations now become

$$\frac{dA(t)}{dt} = A(t)B \tag{5.9}'$$

$$\frac{dA(t)}{dt} = BA(t) \tag{5.10}'$$

$$A(t_1)\,A(t_2) = A(t_1 + t_2)\ ,$$

and of course $A(0) = I$, the identical matrix. Thus, the solution problem is that of a set of homogeneous linear differential equations with constant coefficients, which can always be given in terms of exponentials. Since clearly $A(t)B - BA(t) = 0$ for all values of t, an obvious method of solution is to diagonalize B, etc.

6. Solution of the Stochastic Equations

Having passed the three hurdles--the rational elimination of the human variables, the formulation of the stochastic process, and the obtaining of the transition probability rates--we are left with the practical problem of solving the stochastic equations, essentially (5.4), of the last section. It is submitted that this is a far easier problem than the three former. One may even say that unreasonable difficulties encountered in its handling are usually the consequence of inadequate formulation at the earlier stages.

The stochastic equations (5.4) have the form of the first time-derivative of the unknown, equated to a homogeneous linear functional of the latter. It is a Cauchy problem, i.e., the determination of a function from its initial values. When things are not as simple as in (5.4)', there may be conditions at the boundary or at infinity. In most military problems, there are states into which transitions are irreversible (destroyed units do not come back to life); and absorbing boundaries. These do not cause real difficulties. But with infinitely extended X, there may be a finite probability of a rejection to infinity (cf. the divergent birth processes, Feller, I.e.,

Vol. I, p. 509), or vice versa. In such cases one may introduce an artificial boundary condition preventing this, by making $\hat{X}$ bounded. If the solution of the resulting problem shows little probability of crossing the boundary, the modified version can be accepted as a sufficiently good approximation. If on the contrary there is an appreciable probability that the state will reach the boundary, either the original problem was incorrectly posed, or an important operational reality is being revealed.

Assuming such matters attended to, we list the general methods that can be examined for the solution of the stochastic equations.

A. Formal Manipulations

These always succeed when the equations are of the form (5.9). In more general cases, the linearity of the problem makes the formal methods of the general theory of such equations worth examining, such as changes of variables, separation of variables, Green's function, the method of characteristics. They all have had application to certain special operational problems of the present type. Much information is given in some problems by the equilibrium solution: a function independent of the time.

B. Infinite Series

Again the linearity of the problem makes the expansion in series of the unknown function a simple enough process to be worth a try. For small values of t, power series in this variable can be considered; but more often, expansion in a series of orthogonal functions related to the basic problem offers more promise.

integral transforms, and, in stationary transition cases, Laplace transformation on the time axis, come under this class of method. So also does the method of perturbations.

C. Successive Approximations

This method, sometimes called the Picard process, consists in solving in sequence by integration the recurrent relation obtained from (5.4)

$$\sum_{x} a_{n+1}(x, t; x'', t') = \sum_{x} a_n(x, t; x', t')\, b(x', x'', t')$$

Here, some choice--largely arbitrary--of the initial approximant $a_0(x, t; x', t')$ is made. The other approximants $a_n(x, t' x', t')$ are computed successively for n=1, 2, ... Under very general conditions, the sequence of functions so obtained converges to the desired solution. This method has been used in celestial mechanics for centuries. With the use of modern computers, much of the labor of calculation can be avoided.

D. The Approximation by Difference Equations

The methods most used at present come under this heading. There are two separate steps in the process:

(a) The replacement of continuous time t by discrete period i.e., by "short" time intervals Δt, so that the discrete succession of epochs

$$t_o, t_1, t_2, \ldots ; \quad t_i - t_{i-1} = \Delta t$$

is used, $b(x, x', t)$ is replaced by

$$b_i(x, x') = b(x, x', \bar{t}_i)\,\Delta t \quad , \quad t_{i-1} \leq \bar{t}_i \leq t_i$$

and $a(x, t; x', t')$ by

$$a_{ij}(x, x') = a(x, t_i; x', t_j) \ .$$

Finally, the stochastic equation (5.4) is replaced by the approximate result of integrating it over the interval (t_{i-1}, t_i):

$$a_{j,i}(x, x'') = \sum_{x'} a_{j,i-1}(x, x')\, b_i(x', x'')$$

The fact that by taking the time intervals Δt (which need not all be of the same length) relatively small, a useful approximation can be obtained, is the practical basis of the method, which replaces a differential equation by a recurrence formula. This procedure has long been used in celestial mechanics, and, under the name of Cauchy-Lipschitz method, in the theory of differential equations.

(b) Another step in the process is taken when the number of states x in X forms a continuum, so that the stochastic equations are integro-differential equations, the Σ in (5.4) and (5.5) being actually a $\int$. It is often possible to divide such a continuous X into a finite number of cells, small enough so that the difference in behavior of the system at different states in the same cell can be neglected, yet large enough so that the total number N of cells

covering X can be handled. On labelling each of the cells by a subscript s running from 1 to N, and replacing the integration over X by the approximating summation, the transition probabilities $a(x, t; x', t')$ become replaced by elements $a^{s, s'}(t, t')$ in an N-by-N matrix, and similarly for $b(x, x', t)$ which is replaced by $b^{s, s'}(t)$. Then (5.4) and (5.5) assume the forms (5.4)' and (5.5)' so that the simple theory of such differential equations can be applied. In particular, in the stationary transition case, they become (5.9)' and (5.10)' and can be solved explicitly.

(c) Both Processes (a) and (b) Used Together. The stochastic functional equations then take the form of recurrent matrix relations, relating the transition matrices

$$A_{i,i'} = || a_{i,i'}^{s,s'} || , \quad B_i = || b_i^{s,s'} || .$$

Here $a_{i,i'}^{s, s'}$ is the probability that if the system is in the state of index s at the epoch i it be in the state s' at the epoch i' . Similarly, $b_i^{s, s'}$ is the "elementary" transition probability that if the system has the state index s at epoch i-1, it be in s' at the next epoch i. The stochastic equations are, in matrix form

$$A_{i,i'} = A_{i,i'-1} B_{i'} .$$

These are solved by recurrence, starting from the fact that $A_{i,i} = I$, the identical matrix. The solution is then

$$A_{i,i'} = B_i B_{i+1} \cdots B_{i'}$$

When in particular the process is of stationary transitions, $B_i = B$ (independent of i) and we have

$$A_{i,i'} = B^{i'-i} \quad .$$

If the probabilities $p^{(s)}$ that the system be initially in states s are given, and if P_o is the (horizontal) matrix of these numbers, by the n' the stage they will be replaced by $p_n^{(s)}$; and

$$P_n = P_o B_o B_1 \ \ldots \ B_{n-1} = P_o B^n \ ,$$

the last expression, for the case of stationary transitions.

The replacement of the stochastic equations by difference equations as set forth above under D(c), is the most directly adapted to machine calculation, since all that has to be computed are sums of products of known numbers. The latter can be given in tabular form, no curve-fitting formulae being required. It is of general and uniform applicability, not requiring special conditions or calling for exceptional insights. Finally, it can be described and understood in the terms of elementary mathematics.

The method has, unfortunately, one major disadvantage: in order to keep the number N of cells, into which the actual states in X have been condensed, small enough not to over-run any computer, it may be necessary to make the cells so coarse that the transition probabilities may be quite different for two states in the same cells: then the basic stochastic process is not even approximately Markovian--and the whole logical justification of the computation disappears. Another way of saying the same thing is to

that under such conditions, the difference equations introduced in D(c) are not approximations to the true stochastic equations. Unfortunately, the numbers coming in profusion from the computer cannot be expected to give any warning that the program has lost its logical basis.

A. Monte Carlo Simulations

In this very commonly employed method of studying military operations, data specifying the state of affiars at any epoch of time is programmed into, or produced by, the computer. A rule is also programmed, of such a nature that the computer, when in a particular state at a given epoch, automatically selects a state into which it goes in the succeeding epoch. This programmed selection rule may be deterministic: just one definite state from the preceding state. Or it may be statistical: by the use of a table of random numbers, (mapped in an appropriate way to correspond to a desired probability distribution,) the succeeding state is chosen at random, each possibility with the predetermined probability. It is this "random machine" character of the operation that has given it the name of Monte Carlo.

It should be clear that the machine, when used in this manner, is itself a physical system S; that things have been so arranged that it has a known set X of states x; and that, from epoch to epoch, its states change according to a known program of transition probabilities. Since at any given epoch the Monte Carlo selection of the next state is drawn from a distribution that is determined by the state the system is then in, the transition process is Markovian. Thus, the machine is made to form a system moving from state to state according to a Markov stochastic process. The fact that this system

is regarded as behaving in essentially the same way as the actual military operation has given it the name simulation.

To assume that such a use of machines gives even approximately valid information about the military operation is to assume the following:

1. that the human uncertainties have been removed;
2. that the combat situation involves a system that is, at any time, in a objectively describable state;
3. that its state transitions are Markovian;
4. that its stochastic equations can be satisfactorily approximated by difference equations, as in D(c), without losing their Markovian character;
5. that the repetition of runs gives, by the law of large numbers, satisfactorily accurate and reliable values of the desired probabilities.

Inasmuch as these are precisely the matters examined in succession in the preceding sections of the present paper, it can be said that the logical bases of Monte Carlo simulation have been laid--that it depends for its validity on the reasoning we have been giving.

The question of the cost-effectiveness of its use of machine time, as compared with the use of machines for the direct multiplication of matrices, as described at the end of the preceding section, is a question that is too infrequently raised. In some recent Naval studies the author has found that the direct computation of matrix products has had far greater cost-effectiveness.

The method of Monte Carlo simulation has one particular value: its educative or intuition building effect on those who behold the actual

performance of the process. It allows the results of experimental variations of certain factors of the situation to be perceived in a direct and life-like way. This appearance of realism is so great that it has often led observers to forget that they were not in fact observing Nature directly: a disastrous error.

5. Paths Through the States

Up to now it has been tacitly assumed that the set X of states x (or its coarser finite approximation) could be described individually, so that the transition rates or elementary transition probabilities could be listed, and the stochastic equations--however hard they might be to solve--actually written down. There are enough examples of this situation to lead to such a view. Moreover, it is the typical case in classical mathematical physics.

A few simple problems may force a somewhat brutal change in such an optimistic position.

Consider the combat model of Lanchester, in which, at each time t, there are two forces in opposition, u units on one side and v on the other, the rate of destruction of either being proportional to the number in the other. The system S consists of the two forces. Each state is characterized by the pair of numbers (u, v). If at the start $u = u_o$ and $v = v_o$, the number of states would be $u_o v_o$. For values even as moderate as $u_o = v_o = 10$, this number would be 100 and the transition matrix would have $100^2 = 10{,}000$ elements. In the present case, however, there are two simplifying factors in the situation, which are typical of many cases in which the direct enumerative treatment implied in the preceding sections is impracticable, but in which the solution can nevertheless be given.

The first simplifying factor in the Lanchester model is the circumstance that only a small number of transition rates differ from zero. In a short time Δt, u can go only into itself (probability: $1-bv\Delta t$) or into $u-1$ (probability: $bv\Delta t$). Similarly for v. All other transition probabilities are zero. (All these statements neglect terms of higher order in Δt). Lanchester was led by this fact to use--instead of the stochastic model--a deterministic one, in which (u, v) are regarded as continuous variables having time derivatives given by his well-known equations

$$\frac{du}{dt} = -bv \ , \quad \frac{dv}{dt} = -au \ . \tag{8.1}$$

Here a and b are the "coefficients of effectiveness" of the first and second forces. This has the effect of making all the states into which the system goes, starting from (u_o, v_o), determined by these initial values and the elapsed time t, because of the uniqueness of the solution of the differential equations (8.1). Thus, the problem has the form of problems of classical mechanics.

When, to a good approximation, the transition function $a(x, t; x', t')$ determines the state into which the transition can occur, we say that the system traces out a path in its space of states X. When this is true except at a "small" number of states--nodal states--which allow a "small" number of different states to be entered, we say that we have a graph of paths. When certain of the paths never have any further modal points, they are called branches. There may be cycles, i.e., paths returning to their original state. When the graph of paths has no cycles, it is called (according to the usage in Topology) a tree.

Thus, the reduction of a combat model to a graph of paths, with probabilities entering only at the (exceptional) nodal states, greatly reduces the stochastic complexity of combat models.

How would this simplifying circumstance show itself within the general analytic framework we have been using? The simplest answer is in the case of finite X, when our transition quantities are matrices: Then, except at the columns of transitions out of nodal states, each row in every transition matrix has all but one element zero, the exceptional element being unity. In the continuous case, the formulation would involve the Dirac delta function, but at this point, the usefulness of attaching the special case to the general formulation is largely lost: it is easier and more natural to derive it in the simpler deterministic symbolism in the first place.*

We shall see in §'s 9-11 how the concepts of the graph of paths and the endgame (usually, duel) bring the quantitative study of even quite large-scale military actions into the realm of practicable quantitative treatment.

The second simplifying factor in the Lanchester model is the possibility of expressing the vast number of transition rates by a simple mathematical law--thus avoiding the need of their individual case-by-case enumeration. The habit of individual enumeration is easily developed through association with modern computers, that are so capable of having each special piece of numerical datum programmed into them--whether formulas are available or not. In experience

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*This could be illustrated by Hamiltonian dynamics, in which the state x_t (position and momentum) is a point in phase-space X, determined by its initial value x_0: $x_t = f(t, x_0)$. Ergodic and kinetic theory make use of the probability distribution over X at t, $p(t, x)$. This evolves with increase of t according to the simple equation $p(t, x) = p(0, f(-t, x))$ (in the steady case). This is actually a transition probability (density): $p(t, x) = a(x_0, 0; x, t)$. Yet it would be highly awkward and artificial to write it in the form

$$p(t, x) = p(0, x_0)\ \delta(x - f(t, x_0)) \ .$$

confined to manageable numbers of inputs, the power and flexibility of this method recommend it. But if we have to do this in well over 10,000 cases, it would cease to be a practicable operation.

In the present example, let us write the stochastic equations for the transition matrix $a(x, t; x', t')$--or, because of the steady-transition property, $a(x, t', t'-t)$--as $f(t, u, v)$: the probability that at t the first and second forces have u and v units respectively, given that at $t = 0$ they had u_o and v_o. To terms of higher order in Δt, we have, for small $\Delta t>0$,

$$f(t + \Delta t, u, v) = (1 - bv\Delta t - au\Delta t)\, f(t, u, v) + bv\Delta t f(t, u + 1, v) + au\Delta t f(t, u, v + 1)$$

when $u > 0$, $v > 0$;

$$f(t + \Delta t, 0, v) = f(t, 0, v) + bv\Delta t f(t, 1, v) .$$

$$f(t, \Delta t, u, 0) = f(t, u, v) + au\Delta t f(t, u, 1) .$$

These lead at once to the partial mixed difference equations (Δ_u denoting a u-partial difference $\Delta f(u) = f(u + 1) - f(u)$; and similarly for Δ_v):

$$\frac{\partial}{\partial t} f(t, u, v) = bv\Delta_u f(t, u, v) + au\Delta_v f(t, u, v) \qquad (u > 0, v > 0) \tag{8.2}$$

$$\frac{\partial}{\partial t} f(t, 0, v) = bv f(t, 1, v) \tag{8.3}$$

$$\frac{\partial}{\partial t} f(t, u, 0) = au f(t, u, 1) \tag{8.4}$$

These are the equations of the Lanchester stochastic process, introduced and studied after Lanchester's time by a number of authors during the last quarter century (cf., e.g., Morse & Kimball,"Methods of Operations Research," M.I.T. Press, Cambridge, Mass. (1951). R.H. Brown, "The Theory of Combat: The Probability of Winning," The Jor. of the Op. R. Soc. of America, Vol. 11 (1963) pp. 418-425). By making use of the simple mathematical law of transitions, they remove all the complexity of individual enumeration and reduce the problem to the form of classical analysis.

There are two ways of setting forth the connection between these stochastic equations and the deterministic ones, (8.1). The first recognizes that in the former, u and v are random variables, while in the latter, these symbols denote expected values ($\bar{u}$, $\bar{v}$):

$$\bar{u} = \sum_{u,v} f(t, u, v)\, u \quad , \quad \bar{v} = \sum_{u,v} f(t, u, v)\, v \, . \tag{8.5}$$

Let (8.2) be summed for u fixed but v going from 1 to v_o, and the result added to (8.4). We obtain, since elementary calculation shows that

$$f(t, u, 1) + \sum_{v=1}^{v_o} \Delta_v f(t, u, v) = f(t, u, v_o + 1) = 0 \, ,$$

the result

$$\frac{\partial}{\partial t} \sum_{v=0}^{v_o} f(t, u, v) = b_u \sum_{v=0}^{v_o} v f(t, u, v) \, .$$

Multiply this by u and sum over u from 1 to u_o. On the left we obtain, on comparing with (8.3), $d\bar{u}/dt$. The right-hand member, after writing

$$\sum_{v=0}^{v_o} v f(t, u, v) = \bar{f}(t, u) \, .$$

yields

$$b\sum_{u=0}^{u_o} u\Delta_u F(t, u) = -b\sum_{u=1}^{u_o} F(t, u) + bu_o F(t, u_o + 1) .$$

The last term, involving an impossible value of u, is zero. The first sum on the right can be written as

$$-b\sum_{u=1}^{u_o}\sum_{v=0}^{v_o} vf(t, u, v) = -b\bar{v} + b\sum_{v=0}^{v_o} vf(t, o, v) .$$

Now if u_o and v_o are substantially greater than zero and if t is moderate, the last sum has all $v_o + 1$ of its terms exceedingly small, since this is the case of f(t, o, v). Thus, we are led to the first equation in (8.1) for $(u, v) = (\bar{u}, \bar{v})$ as a good approximation. Similarly for the second.

The second method of connecting (8.1) with (8.2) - (8.4) consists in replacing the differences by their approximations as derivatives times the length of the interval (unity)--deliberately ignoring the fact that f(t, u, v) had originally been defined only for integral valued u, v. We obtain from (8.2) the homogeneous linear first order partial differential equation in f = f(t, u, v)

$$-\frac{\partial f}{\partial t} + bv\frac{\partial f}{\partial u} + au\frac{\partial f}{\partial v} = 0 . \tag{8.6}$$

This is easily solved by the method of characteristics, the equations of which are

$$\frac{dt}{-1} = \frac{du}{bv} = \frac{dv}{au} \tag{8.7}$$

but these are the Lanchester equations (8.1). Their "fundamental system" of integrals can be found explicitly by elementary formulas, and then the solution

f of (8.6) is the function of them chosen to fit the initial conditions: to reduce to a given function $f_o(u, v)$ when $t = 0$. In the usual case, $f_o(u, v) = \delta(u - u_o)\ \delta(v - v_o)$. We will illustrate this by finding the curves in the (u, v)-plane into which the paths (in (u, v, t) -space) project. We have but to find the time-independent integral; i.e., to integrate the last equation. It gives at once

$$au^2 - bv^2 = au_o^2 - bv_o^2 ,$$

i.e., a family of hyperbolas, which are the level surfaces of any time-independent solutions of (8.6): these are the geometric paths, traced out by our system with passage of time.

It must be emphasized in conclusion that the deterministic path equations (8.1) are only an approximate rendering of the stochastic equation of the Lanchester process (8.2) - (8.4), acceptable only under special conditions. The fact that such graph-of-paths simplifications are possible, as well as the fact that they are never more than approximate, are typical of a host of operational problems of this category.

9. Patterns of Flow

The points made in the last secion, and others as well, are illustrated by a particular operation that extends the Lanchesterian one by requiring that destruction must be preceded by detection.

Let the two forces engaged in combat be composed initially of u units on one side and v on the other. Let it be assumed that any unit not

in contact with hostile ones must first detect a hostile unit and will then attempt to kill it. By "detect" we mean not only perceive the presence of, but identify and localize sufficiently for attack.

What are the possible states of such a system?

Evidently each individual unit on one side can be either dead or alive; in the latter case, his state--in the Markovian sense, of determining the chances of what may happen to him--depends on the set A of hostile units he is detecting and the set B of hostile units that have detected him. But of the set A, some may be detected by the unit's friends; and similarly for B. The outcome of the combat may be supposed to depend also on the states of the friendly units in contact with A and B. Clearly, the full set of relationships is not simple, even to formulate. Let us give an indication of the possible number of states that may come into play.

At any time, let there be m and n units alive on the two sides. Indicate the units by m dots on the right, n on the left. Indicate that a right-hand unit has detected a left hand by drawing a blue line connecting the corresponding dots; and join with a red line two opposing dots to show that the one on the left has detected one on the right. The resulting colored graph determines the state of our system. How many different graphs are possible? Of the mn possible ways of drawing the blue lines, any one can actually be drawn or not. Hence, there are 2^{mn} possibilities for the blue lines; and similarly for the red. Consequently, there are 2^{2mn} possible colored graphs. If there were m = 5 and n = 4, and if the units were all individually different in their characteristics, a realistic account might

have to consider 2^{40}, or over 10^{12} different states, with over 10^{24} elements in the transition matrix (more molecules than in a liter of gas).

The number is considerably reduced when all units on each side are identical. To estimate the number of states, we note that any permutation of the right-hand dots followed by a permutation (in general, different) of the left-hand dots takes a colored graph into a colored graph, either identical with the original one, or else different, but of the same type (not only topologically, but having the same probabilities of transitions). Let there be a totality of k different types of graph, and, after giving each type a subscript in an arbitrary way, let there by N_i different graphs of the i'th type. Since there are in all m! n! double permutations, evidently $N_i \leq m!\, n!$. On the other hand, the sum of N_i for i = 1, ..., k is the quantity 2^{2mn} encountered before. Putting these facts together, we obtain

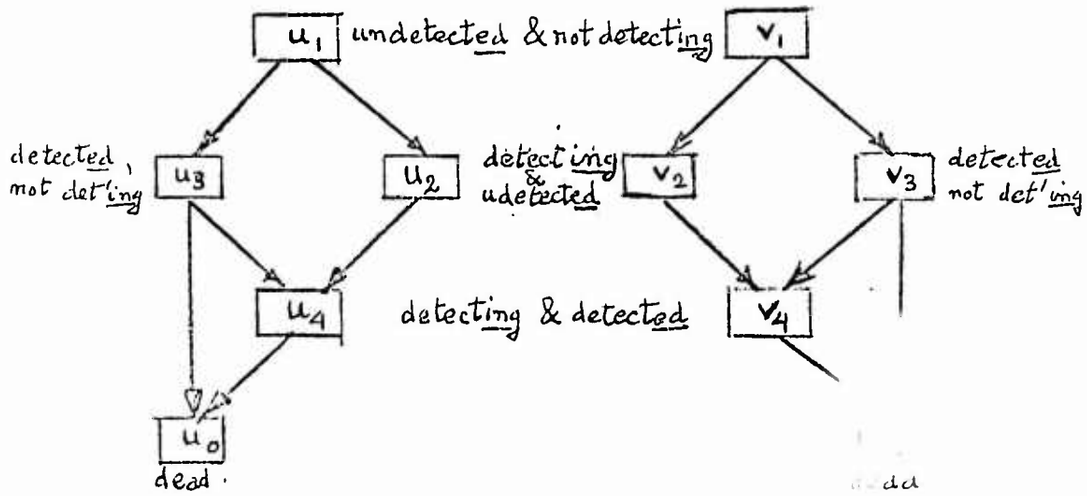
$$k \geq 2^{2mn}/m!\, n!$$

a quantity well over the hundred millions in the relatively simple case of m = 5, n = 4. Evidently, it would be useless to await the advent of larger and faster computers into which these states--and their transition matrix--can be programmed.

Let us see if certain aspects of the detection-destruction engagement can be isolated from the welter of possible interactions suggested above--just as hydrodynamic relations can be isolated from the unthinkable complexities of the molecular motions in a fluid.

Let us imagine five boxes on the left and five on the right (Fig. 1). All the units on the left side are represented as particles in one of the five

Fig. 1 - The Flow Graph in a Detection-Destruction Combat



boxes on the left, according to their live-dead, detecting-nondetecting, detected-nondetected status, as indicated. Similarly with those on the right. If the total number of units, dead or alive, on the left is u, and on the right is v, the total number of ways is the number of partitions D_5^u of the positive integer u into five parts (the number of ways that five people can be paid with u indistinguishable coins), times (almost) the corresponding number D_5^v. It is known that $D_r^u = C_u^{u+r-1}$, the binomial coefficient*. Hence, the number of ways the boxes of Fig. 1 can be filled is almost--but not quite--

$$C_u^{u+4}\, C_v^{v+4} = \frac{(u+1)\,(u+2)\,(u+3)\,(u+4)\,(v+1)\,(v+2)\,(v+3)\,(v+4)}{41 \cdot 41} ,$$

* An easy way to show this is to write the geometric series $1/(1-t_i) = 1+t_i+t_i^2+\ldots$ Take the product over $i = 1, 2, \ldots, u$. On the right we have D_r^u terms of degree r, yielding the term $D_r^u t^r$ when every t_i is replaced by t. But this substitution replaces the product on the left by $(1-t)^{-u}$, whose binomial expansion yields $C_u^{u+r-1} t^r$ as the term of degree r.

When u = 5, v = 4, this number becomes nearly (126) (70) = 8,820, an uncomfortably large number with a $(8,820)^2 > 10^7$ element transition matrix; but still incredibly less than in the earlier case.

Two points must be considered: First, is the reduction from the former case to the present one acceptably realistic, or is something essential lost in the passage, so that fatal misconceptions will result from the simplification? Second, granted that the description embodied in Fig. 1 is acceptably realistic, how can we manage to handle even this relative simplification?

The answer to the first question will not be given here. It will depend on the situation studied and the purpose of its study. But we insist on the fact that whenever a war game is programmed on a computer, an affirmative answer to this type of question is implied logically--even though only unconsciously.

We shall deal with the second question. Fig. 1 not only shows the boxes of individual states, but, by connecting arrows, the possible transitions as well. These are based on the fact that a unit has a chance of being killed if and only if it has been detected.

Let us think of the units in each box as merged into a sort of fluid flowing as a fluid through the connecting lines (pipes) in the direction of the arrows. We must establish a rate of flow through each pipe.

This brings us to the mechanism whereby detections and kills are made. We shall assume--as descriptive of many, but not all, cases that if the number of undetected units on one side is doubled, the detection probability of each unit is halved, so that the total expected number detected in the

next short period Δt is left unchanged. We shall also assume that this total expected number is proportional to the number of enemy units that are alive but have not detected any unit (whether or not they are themselves detected). This assumption--quite different from those made in chemical kinetics--reflects the character of the act of detection in the present battle: it involves localization and approach, and so occupies the unit with a single enemy unit to the exclusion of all others. Accordingly, the flow:box 1 to box 2,shall be regarded as proportional to $v_1 + v_3$; i.e., as equal to $b_1(v_1 + v_3)$. The constant of proportionality b_1 is a figure of effectiveness of detection per unit on the right ($\underline{b}$ will denote effectiveness coefficients of the right; $\underline{a}$, for the left).

Concerning the kill rates, one might consider the Lanchesterian assumption that the expected number of left-hand units killed is proportional to the number of right-hand units that have detected them. For two reasons, such an assumption cannot be applied to the present situation. The first reason is the practical one that, within the framework of Fig. 1, we do not know how many of the $v_2 + v_4$ detecting units on the right are detecting the v_2 units in box 2 on the left and how many, the v_4 in box 4. To try to specify this would lead us back to the earlier case, the simplification of which was the object of the introduction of the diagram of Fig. 1. The second reason resides in the conception of the present type of engagement, where, if detection is carried out on an individual basis, so is a subsequent kill.* All this, of course, is for an elementary time interval Δt, so short that only one thing has any likelihood of happening. Thus, we shall assume

* As when two friendly units never engage the same enemy unit, fearing mutual danger of own weapons — and so intercommunicate.

that each detected unit on the left has a chance b_2 of being killed which is independent of the possible number of additional units on the right. Thus the expected number out of n detected units that are killed during Δt is $b_n n\Delta t$, to quantities of higher order in Δt. Accordingly, the flow rates along the pipes from boxes 3 and 4 into 0 on the left are $b_2 u_3$ and $b_2 u_4$, respectively.

The rates of flow which reflect the conceptions just set forth are given diagrammatically in Fig. 2, which assumes that no unit playing an essential part is absent. It is given for the left-hand side of Fig. 1. That for the right-hand side is the same with every u and v interchanged and every a and b interchanged, subscripts remaining unchanged.

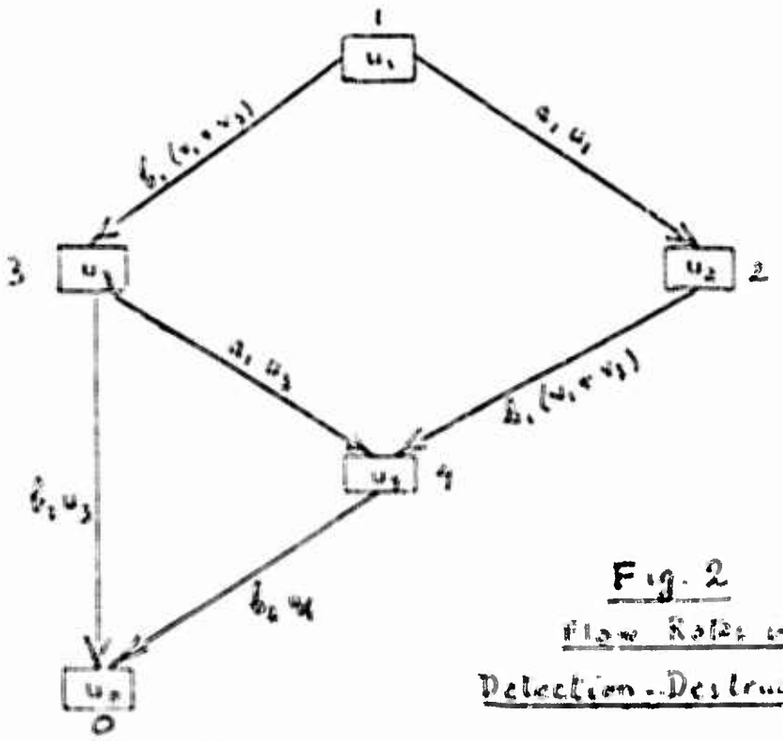


Fig. 2
Flow Rates in the Detection-Destruction Combat

Assumptions

From these, by applying an obvious conservation principle, we obtain the differential equations (in addition to the relation $u_0 + u_1 + u_2 + u_3 + u_4 = \text{constant}$):

$$
\begin{aligned}
\frac{du_1}{dt} &= -a_1 u_1 - b_1(v_1 + v_3) \\
\frac{du_2}{dt} &= -b_1(v_1 + v_3) + a_1 u_1 \\
\frac{du_3}{dt} &= -(a_1 + b_2)u_3 + b_1(v_1 + v_3) \\
\frac{du_4}{dt} &= -b_2 u_4 + a_1 u_3 + b_1(v_1 + v_3)
\end{aligned}
\tag{9.1}
$$

There are four more equations obtained from the above by the interchange of u and v , a and b.

Thus, our deterministic flow replacement of the stochastic process leads to a homogeneous linear system of the eighth order, with constant coefficient. It can be solved explicitly in terms of exponentials, and the evolution of the combat completely dominated.

This is the deterministic or flow treatment of the detection-destruction process in simplified form. A stochastic treatment could be given, introducing the function

$$
f(t;\, u_1, u_2, u_3, u_4;\, v_1, v_2, v_3, v_4) \tag{9.2}
$$

which is the probability that at the epoch t the state of the system be as indicated by the (u, v) letters. It is easy but lengthy to write its stochastic

differential-difference equation similar to (9.2), etc., and to derive the (9.1) as the equations of the characteristics of the approximating partial differential equation.

It is emphasized that all the present deductions from Fig. 2, in particular, equations (9.1), are only valid in the region in the eight-dimensional space of the positive variables (u, v) for which the inequalities of Fig. 2 hold.

10. A DETECTION-DESTRUCTION DUEL

The detection-destruction engagement just examined becomes simple enough to solve analytically without approximation in the case in which it reduces to a duel: only one unit on each side. Then the set of states is representable by the diagram of Fig. 3, in which, if we call the the two opponents the "left" and "right" units (people, aircraft, naval craft, etc.), we have numbered the states as follows:

State 1, neither unit detecting the other;

State 2, left detecting right, right not detecting left;

State 3, left not detecting right, right detecting left;

State 4, each detecting the other;

State 5, left dead, right alive;

State 6, left alive, right dead.

Since pairs of transitions have a probability of higher order than Δt of occurring during this interval, the stochastic equations will give a zero rate of change the probability that both be dead. Therefore, this state cannot be entered in the present model (it can be, in many other models). Consequently,

it is omitted here. We are denoting the transition rates expressing the left combattant's effectiveness by a, those of the right by b. Thus, a_1, a_2, a_3 are the rates of detection, kill in states 2 and 4, by left against right, etc.

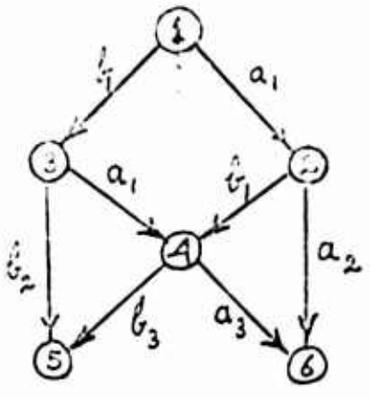


Fig. 3
Transition Rates in the Detection - Destruction Duel

This schema of relations is equivalent to the following transition rate matrix, B of §5, appearing in (5.9)'.

$$B = \begin{Vmatrix} -a_1-b_1 & a_1 & b_1 & 0 & 0 & 0 \\ 0 & -a_2-b_1 & 0 & b_1 & 0 & a_2 \\ 0 & 0 & -a_1-b_2 & a_1 & b_2 & 0 \\ 0 & 0 & 0 & -a_3-b_3 & b_3 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

We now apply (5.9)' to this case, writing for brevity $a(x^o, 0; x, t) = P_x(t)$ (the probability that at $t>0$ the system be in the state x, having been in x^o at $t=0$), and denote the time derivative by $P'(x)$. This leads to the

stochastic equations (which might have been obtained intuitively from a flow picture based on Fig. 9):

$$
\begin{aligned}
P_1'(t) &= -(a_1+b_1)P_1(t) \\
P_2'(t) &= a_1P_1(t) \quad -(a_2+b_2)P_2(t) \\
P_3'(t) &= b_1P_1(t) \qquad\qquad -(a_3+b_3)P_3(t) \\
P_4'(t) &= \qquad b_2P_2(t) + a_2P_3(t) - (a_4+b_4)P_4(t) \\
P_5'(t) &= \qquad\qquad b_3P_3(t) + b_4P_4(t) \\
P_6'(t) &= \qquad a_2P_2(t) + a_4P_4(t)
\end{aligned}
\tag{10.1}
$$

These equations can be solved explicitly and individually, starting with the first, and working down the list in the order in which they are written. The most interesting case is that in which the system is initially in State 1 (both undetected): $P_1(0) = 1$, $P_2(0) = \ldots = P_6(0) = 0$. Then as time increases, the first four probabilities, after becoming positive, later approach zero exponentially, whereas the last two approach positive limits $P_5(\infty)$ and $P_6(\infty)$ (adding up to unity), interpretable as the probabilities of victory for the right and left, respectively. While these form a steady state solution, their values cannot be calculated by simply replacing all the left-hand members in (10.1) by zero--a commonly used method in the study of stochastic processes.

Solving (10.1) as indicated above, and then letting $t \to \infty$, we obtain (after elementary although perhaps tedious reductions)

$$
P_5(\infty) = \frac{b_1b_2}{(a_1+b_1)(a_1+b_2)} + \frac{(a_1+a_2+b_1+b_2)a_1b_1b_3}{(a_1+b_1)(a_3+b_3)(a_1+b_2)(a_2+b_1)} \tag{10.2}
$$

visibly equal.

$$P_{(0,0)} = \frac{a_1 a_2}{(a_1+b_1)(a_2+b_2)} + \frac{(a_1+a_2+b_1+b_2)\,a_1 b_1 a_2}{(a_1+b_1)(a_2+b_2)(a_1+b_1)(a_2+b_2)}$$

A few facts can be immediately inferred from these equations. For example, if the detection rates a_1 and b_1 are much smaller than the other quantities, the probability of one side winning is to that of the other as the detection rate of the former is to that of the latter; detection is all the duel. If, on the other hand, the two detection rates are much larger than the kill rates, so that the last term in each equation in (10.1) is much greater than the ones before, the win probability ratio is as the rate of detected kill rates $a_1 : b_1$. There are a few other rather obvious deductions, in cases that some of the transition rates are much larger or much smaller than others, derivable by manipulating (10.2), and which, when once found, seem self-evident.

The most obvious general deduction from (10.2), however, is in the case of nonextreme values of the constants: that no easy "common sense" judgment would have been likely to have drawn simple conclusions from the detection-destruction duel. Inasmuch as the present one is about as simple a realistic model as one could set up, this conclusion applies a fortiori to any duel. The complexities involved, expressed here by equation (10.2), result from the complicated quantitative interplay of sightings, kills, and their mutual forestallings, that are all in the actual nature of the case--not an artifact of the technique for its investigation.

[illegible] method, on the other hand, the derivation of (10.4) from [illegible] directly by actually solving (10.1) and then letting $t \to \infty$ [illegible] employed [illegible] simple case, and [illegible] possible in more complicated cases when explicit solutions [illegible] be [illegible] difficult. The method is the one familiar to physicists and [illegible] of taking the [illegible] transform [illegible] then integrating from 0 to ∞. Thus $\psi_n(t)$ is replaced by its [illegible] transform

$$\varphi_n(s) = \int_0^\infty e^{-st} \psi_n(t)\, dt .$$

Use is made of the formula obtained by integration by parts

$$\psi_n(0) = s\varphi_n(s) - \int_0^\infty e^{-st} \psi_n'(t)\, dt .$$

This transformation has the effect of algebraizing the problem, so that it is free from derivatives. Moreover, the solution of the resulting six linear equations in the six functions $\varphi_n(s)$ does not require the solution of any algebraic equation of degree higher than the first. When they have been solved, the values of $\psi_n(\infty)$ are obtained by multiplying the results through by s and letting $s = 0$, use being made of the equation

$$\lim_{s=0} s\varphi_n(s) = \psi(\infty)$$

which is the simplest case of an Abelian theorem and is provable by simple limit reasoning based on Abel's lemma and the second law of the mean.

The method can, of course, be applied for any initial values. Results as simple as (10.3) but different are obtained, and thus the advantage of [illegible] investigated.

However the derivation, the question of the probabilities of winning, and how they are changed by changing tactics or equipment, has to be answered by actually substituting the numbers into the equations and calculating the results.

[illegible]

We have seen that, once the effects of human unpredictability have been aggregated, every combat model, Monte Carlo simulation or program, has at its core a simple structure of a universal type: a system and its states, and a law of transitions (deterministic or probabilistic) governing the change of states in the evolution of the system with time. This is the basic stochastic process of the model. We have seen further that it has always in fact been a practical necessity so to conduct the analysis or the computational manipulations that the statistical quantities desired for a decision are obtained by an iterative step-by-step build-up from the simpler transition probabilities or rates: the stochastic process is treated as if it were Markovian. Logic does not exclude the contrary assumption; but this would involve making every elementary transition depend on a good part of the past history of the system, thus complicating the situation beyond feasibility.

We have seen that the Markov process of the combat model is dominated by the stochastic equations, which express implicitly the way in which the

[illegible] criteria for decisions on tactics of weapon systems are determined [illegible] input data, which express themselves in terms of the elementary [illegible] of states. Finally, we have seen how their solution can be carried [illegible] by a variety of analytical or computing mathematical methods, once the input data are known, provided that the stochastic equations can be written down.

We have seen that the latter proviso, which is no problem in classical [illegible] of applied mathematics may be the central stumbling block in the [illegible] model: even after accepting considerable simplifications and approximations, the number of items to be written down or programmed in a machine may exceed the number of molecules in a liter of gas under standard conditions. This is in part because, even if each unit involved in the system has a small number of states when viewed alone, the number of inter-relations between the set of units may be enormous. In part it is because each unit can itself be in a complex set of states. Thus, if a naval craft has to track an enemy unit during an appreciable interval of time, its phase, or the age of its tracking situation, is required in order to specify its state in the markov process. This is a continuous parameter, which, when replaced by a finite set of intervals in a computational approximation, would lead to a considerable increase in the number of states for that unit. Even if there were no complex of interconnections of many systems, the fact that a system composed of n similar units each having s possible states would be a system having about s^n possible states, and a transition matrix of s^{2n} elements, shows how easily the set of terms or numerical items can grow out of control.

Even if, by some magic, such great multitudes of items could be written down and handled numerically, the accuracy with which each one is known is limited. If the inaccuracies are regarded as small random errors, their combination in large numbers easily leads to a large statistical factor in the final answer. Thus, in a certain study involving billions of dollars over [illegible] statistical entries were required. A simple order of magnitude estimate was that an individual percentage error of about ten percent might easily be multiplied by $\sqrt{3000}$ or about 50, so the answer could be affected by a factor of 5. Unless such a possibility were excluded, the answer could not be applied realistically.

We have seen in sections 8 and 9 how to replace a model of a unit-unit action by another model, intended to give only the general course of events as a graph having a small number of branches, forming a flow diagram that can be analyzed numerically. Of course this gives up the study of detailed interactions; but it may retain considerable realism, as when each of the opposing forces remains together and the interactions involve chiefly detection and approach of the two forces. When after such preliminary operations, the actual combat takes place between pairs of individuals (the duel), or in very small groups, the more detailed study of such actions can often be made using the stochastic equations, as in §10. Such operations have been called endgames by G. Raisbeck, in analogy to the closing stages in chess; although after the military duels, surviving units may reenter the larger game.

For the most part, the effects of the proposed weapon systems or tactics under study are revealed by the endgame. But some may already be

or in the largest scaled flow graph. But in all cases, this grossest treatment is useful in helping to see just what endgames--and consequently, what tactical or weapon factors--do or do not present themselves, and in what proportion of cases. It is, therefore, a guide to relevance. It would, for example, exclude building massive pillows on naval units for the purpose of fending off hits, not because such effectives would be ineffective in endgames involving direct contact of the units, but because such endgames would not occur; they would not terminate any branch of the flow graph followed with any appreciable probability. Even when a costly improvement does occur, if its chance of occurrence is slight, this has to be entered into the data in any cost-effectiveness investigation.

In conclusion, we may say that the logical quantitative study of combat actions is a <u>necessary</u> condition for any realistic study of military cost-effectiveness. Of course it is not a <u>sufficient</u> condition: economic factors going far beyond the present investigation must intervene.

N/A

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When a probability distribution of a target is given, the problem of detecting it by a given method involving the progressive expenditure of effort, will be harder or easier according to the nature of the distribution. A natural measure of the degree of difficulty of detection by the method in question is the expected value of the quantity of effort used up to the (random) moment of detection--assuming that the most effective use of the effort is made at each moment.

The first object of this investigation is to show the connection between the present concept when the search is done as a procedure of successive dichotomic ... localization--to a pre-stated order of precision--is obtained. The connection with the classical theory of information (entropy and cross-entropy) is shown.

The second object of this investigation is to apply the method to the case in which the law of random search is applied progressively and optimally to the search for a target in a linear, planar, or spacial region (or in r-space).

(UNCLASSIFIED)